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AN ELEMENTARY

TREATISE ON MECHANICS.

PART I.—STATICS.

WORKS

BY THE

REV. ISAAC WARREN, M.A.

(FORMERLY MATHEMATICAL SCHOLAR).

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Elements of Plane Trigonometry;

A Table Book of the Weights and Measures used in the British Empire according to the most recent Acts of Parliament;

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PREFACE.

THIS ELEMENTARY TREATISE on STATICS is the first part of a Work on MECHANICS, in the second part of which DYNAMICS (used by the Author to include Kinematics and Kinetics) is treated. In this volume an experimental proof *alone* is given of the Parallelogram of Forces; other proofs of this important proposition will be found at the end of Part II. The Book is constructed on somewhat the same lines as the Author's "Elementary Treatise on Plane Trigonometry," which has now reached a Third Edition. The Book will be found rich in Exercises—a feature which the Author hopes will recommend it to Practical Teachers.

ISAAC WARREN.

11, TRINITY COLLEGE, DUBLIN.
May, 1889.

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AN ELEMENTARY

TREATISE ON MECHANICS.

PART I.—STATICS.

CHAPTER I.

INTRODUCTORY.

1. **Mechanics** is the Science of Rest, Motion, and Force.

2. **Matter.**—We give the name Matter to that which exists in space, and manifests its presence by the possession of such qualities as Extension, Resistance, Impenetrability, &c.

3. **A Body** is a portion of matter limited in every direction.

4. **Rigid Bodies** are such that their parts under all circumstances preserve constant distances from each other. They are hence incapable of deformation (*vide* Note at end of Chapter II.).

5. **Particles** are indefinitely small Rigid Bodies, so that all Forces (Art. 11) acting upon them meet in a point.

We may consider Particles as Material Points.

Throughout this Treatise Body means Rigid Body, unless otherwise stated.

Every body may be conceived to be made up of an indefinitely great number of material particles.

6. **Mass** is the quantity of matter a body contains.

7. **Volume** is the amount of space a body occupies. Volumes are expressed in cubic inches, cubic centimetres, &c.

8. **Motion** is change of place. The opposite of motion is **Rest.** We consider a body to be at rest when it does not change its place with regard to surrounding objects; in motion when it does.

9. **Kinematics** is the consideration of motion apart from the causes which generate them. It is therefore a branch of Pure Mathematics. In this treatise Kinematics is included under Dynamics.

10. **Kinetics** is the Science of Force (Art. 11) in relation to the motions it produces.

11. **Force** is any cause which produces, destroys, or alters the motion of a body, or which tends to do so.

12. **Force must Act upon Matter.**—It is erroneous to speak of Force as acting *on* a Mathematical Point.

Whenever Force manifests itself there is evidence of Matter both acting and acted upon. We may, however, say that Force acts *on* a *Material* or *at* a *Mathematical* point.

13. **Equilibrium.**—If one force *alone* act on a body it must either produce motion or change of motion. When two or more forces act on a body they may neutralize each other's effects, in which case they are said *to balance or equilibrate.* Forces then are in Equilibrium which, when acting on a body, do not alter its state *either of rest or uniform motion in a right line* (*vide* Art. 32).

14. **Mechanics divided into Statics and Dynamics.**—Statics treats of balanced Forces; Dynamics, of unbalanced Forces, and their effects (*i.e.* motions). The term Dynamics in this Treatise is used to include Kinematics and Kinetics (Arts. 9, 10).

15. **Statical Measure of Force.**—A Force is estimated Statically by the weight which it is *just* capable of sustaining vertically (*i.e.* against gravity).

16. **Gravitation Units of Force.**—When a Force is estimated Statically it is expressed in some unit of weight (such as pounds or kilogrammes). Such Units of Force are indefinite, inasmuch as Gravity is not the same over the whole surface of the earth. As long, however, as we keep to one place they are sufficiently definite.

17. **Force considered Dynamically.** — Dynamics (more correctly Kinetics) treats of unbalanced Forces, *i.e.* of Force producing or altering motion; and it is most important to observe that when an unbalanced Force acts on a mass of matter *it must* either produce or alter motion. For example, if a mass of 100 tons were placed on a perfectly smooth horizontal plane, and were pressed parallel to the plane by a force of a single grain, the mass would begin to move; and if the pressure were continued its motion would be continually accelerated.

18.* **Force estimated Dynamically.**—In Dynamics (more correctly Kinetics) Force is considered as a generator of momentum (product of mass and velocity), and Forces are compared by the quantities of momentum they can generate in the same time.

19. **Dynamical Units of Force.**—The Dynamical Unit of Force is that force which produces unit velocity in unit mass in unit time, *e.g.* if

1 lb. = unit mass,

1 sec^{d} = unit time.

1 foot = unit length;

then Unit Force = that force which produces a velocity of 1 foot per second in 1 lb. mass in one second. This force is called a *Poundal.*

---

* To understand this and the next Article fully, Part II., Chaps. iii. and v., must be read.

In the C. G. S. system, *i.e.* where

Unit length = centimetre,
Unit mass = gramme,
Unit time = second;

Unit Force = that force which produces a velocity of 1 centimetre per second in a gramme mass in one second = a *Dyne*.

This subject is more fully considered in Part II. (Dynamics).

20. **Ambiguity of the Word Pound.** — The word Pound is ambiguous, sometimes being used to signify a *mass of matter;* at others, *the weight* of that mass. A pound of mass is an invariable quantity of matter: its weight is variable, and dependent on locality.

The standard Pound is a piece af Platinum prcserved in the Exchequer Office, London. It contains a quantity of matter = $\frac{1}{16}$ gallon (277·123 cubic inches)* of distilled water at 62° F. The $\frac{1}{7000}$ lb. = a grain.

The Pound weight is the *force* with which the standard Pound mass is attracted towards the Earth's centre, and evidently varies with that attraction.

If the standard Pound mass were taken to the surface of the Sun, its weight would be enormously increased, but its mass would remain unaltered. A similar ambiguity embarrasses all gravitation units of force, and hence the necessity for estimating force Dynamically.

21. **Statical and Dynamical Forces Identical.**— There are *not* two kinds of Force. The proper measure of *every* Force is the momentum it generates in unit time; but by expressing Force in gravitation units we are enabled to postpone the idea of momentum till a later stage of our inquiry. Momentum is considered in Part II. Chaps. iii. and v.

---

* Formerly the gallon was *defined* to contain 277·274 cub. in.; but on a more accurate determination of the density of water, this alternative part of the definition was repealed.

22. **The Spring Balance** consists of a spiral spring, which is elongated or contracted by the *force* to be estimated. Its efficiency depends on the torsional rigidity of the wire composing it. By means of this instrument we can measure force as distinguished from mass (estimated by common balances), for its elongation or contraction depends on the *force* which affects it; and hence the same body will produce different effects upon it, according as the instrument is used at different places on the Earth's surface.

23. **Time, how Measured.**—An ordinary clock, when perfect, shows mean solar time. A mean solar day is the time between two successive noons as shown by such a clock. It consists of 24 hours, 1440 minutes, or 86,400 seconds. Any of these denominations may be taken as the unit of time. A second is that usually taken.

24. **Space, how Measured.**—One foot is the unit of length generally used in England. It is the 3rd part of the *standard yard*, which is the distance between two gold plugs in a bronze bar preserved in the Exchequer Office, London, when the whole has a temperature of 62° F.

25. **The F. P. S. System.**—Since force is measured by momentum generated in unit time, and since momentum involves mass and velocity, and therefore space mass and time, it is necessary to have units of these quantities. In the British system of what are called *absolute units*, feet pounds and seconds are the units respectively adopted. Hence the name F. P. S.

26A. **The C. G. S. System.**—Here the centimetre, gramme, and second, are taken as the units of space, mass, and time, respectively.

The Metre is the standard unit of length, and is defined to be the distance between the ends of a platinum bar (preserved at Paris), when at the temperature of melting ice. The Metre was originally defined to be the ten millionth part of a

quadrant of the Earth's meridian through Paris. The Metre is divided into 10 decimetres, 100 centimetres, &c.*

The Gramme = unit of mass = quantity of matter in 1 cubic centimetre of water at 4° C.

An account of this and other units will be found in App. II., Part II.

26B. **The Density of a Body** is the number of units of mass contained in unit volume. The density of water is, in the C. G. S. system, one gramme to the cubic centimetre; in the F. P. S. system, about 62·5 lbs. to the cubic foot.

The Specific Gravity of any Substance

$$= \frac{\text{weight of any volume of substance}}{\text{weight of an equal volume of the standard substance}}.$$

The standard substance is water for solid bodies, and atmospheric air or hydrogen for gases.

QUESTIONS ON CHAPTER I.

1. Define Force.
2. What is the proper way to measure Force?
3. How is Force measured statically?
4. What is a Pound?
5. What is a Poundal?
6. What are the Units employed in the British system of absolute units?
7. What are the Units employed in the C. G. S. system?
8. How was the Metre originally, and how is it now, defined?
9. How is the Standard Yard defined?
10. What is the use of a Spring Balance?
11. What is the weight of a Gallon of Water? *Ans.* 10 lbs.
12. How many cubic inches are there in a Gallon Measure?
Ans. 277·123 cubic inches.

* For an account of the Metric System of weights and measures, see Warren's *Table Book* (Longmans, Green, & Co., London).

CHAPTER II.

DEFINITIONS AND AXIOMS.

27. **Specification of a Force.**—Four things are necessary to completely specify a Force—

(1) Its magnitude.

(2) Its point of application.

(3) Its line of action.

(4) Its direction or sense, *i.e.* from right to left, or left to right, along line of action.

The student must observe the distinction between line of action and direction. Thus in Fig. 18, Art. 73, P and Q act in the *same* direction along *different* lines of action.

28. **Representation of Forces.**—Forces may be represented by straight lines, marked with arrow-heads, to indicate direction; for the four particulars of the last section can be all represented by arrowed right lines.

To indicate magnitude we must select some unit of length and some unit of force, and draw the representative line so as to contain as many units of length as the force contains units of force.

Example.

Represent a Force of 3 lbs., acting at the middle point (M) of a line AB, and making with it an angle of 30°. At M draw MX, making $\angle BMX = 30°$. From MX cut off $MF = 3$ linear units (say inches). Then MF represents 3 lbs. Observe *MF represents* the Force, but *is not* itself a Force.

29. **Proper Representation of a Force.**—The representative line must be drawn *from*, not *to*, the point of application. In Fig. 1, QO is not a proper representation of a Force acting on O, but OQ'.

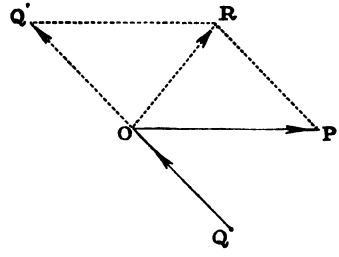


Fig. 1.

30. **Resultant—Components.**—When the joint effect of a set of forces acting on a rigid body can be produced by a single force, this single force is termed the Resultant of the Partial Forces; and, relatively to the Resultant, the partial forces are termed Components. In Fig. 1, OR is the Resultant of OP and OQ', which latter are Components of OR.

31. **Composition and Resolution of Forces.**—*Composition* of Forces takes place when two or more Forces are replaced by a single Force equivalent to them; *Resolution* when the components are substituted for their resultant.

32. **Equilibrium.**—A body is said to be in Equilibrium when it remains at rest. A set of forces are in Equilibrium when they neutralize each other's effects. When a set of forces are in Equilibrium their resultant = 0 (*vide* Art. 13).

33. **Axioms.**—

(1) Forces acting on a particle have a resultant.

(2) If resultant = 0, forces are in equilibrium.

(3) Resultant of a set of forces acting along a right line = their algebraic sum.

(4) The resultant of two intersecting forces falls between them (in the non-re entrant angular region).

(5) The effect of any system of forces acting on a rigid body is unaffected by the introduction or removal of any set of forces in equilibrium among themselves.

N.B.—Forces acting in one direction along a right line being +, those in the opposite direction are −. By Axiom 2, P and $-P$ along same right line balance.

34. **Two intersecting Forces in Equilibrium are equal and opposite.** For if not, they have a resultant acting between them (Axiom (4)), and are therefore not in Equilibrium, which is absurd.

35. **Any Force of a Set of Forces in Equilibrium is equal and opposite to the Resultant of the remaining Forces.**—For replace the other Forces by their Resultant. This Resultant and the Force in question are in Equilibrium, and are therefore equal and opposite (Art. 34).

36. **Transmissibility of Force.**—When a Force acts on a rigid body, the effect of the Force will be unchanged at whatever point of its line of action it be applied, provided the point be a point of the body, or be rigidly connected therewith.

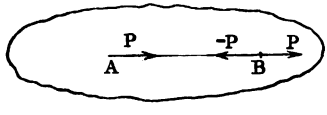


Fig. 2.

Let P act at A. Introduce P and $-P$ at B (Axiom 5) (Art. 33); then P at A, and $-P$ at B, balance (Axioms 2 and 3), leaving P at B; ∴ &c.

Cor. 1.—If a force can be transferred from a point A to a point B of a rigid body without altering its effect, AB is its line of action.

For AB is a line of action, and it is evident there can be only one.

37. **Ways in which Force may be Exerted.**—In what follows, four ways in which force is exerted will come under our notice: 1°. Attractions. 2°. Pressures. 3°. Reactions. 4°. Tensions. We shall consider each of these briefly.

38. 1°. **Attraction** is exerted without the intervention of any visible instrument, as when a magnet attracts a piece of soft iron.

All matter is endowed with the power of attracting all other matter, and it is to this power that bodies owe their weight. This force is called the Attraction of Gravitation. The attractive force between two masses, m and m', at distance d, is given by the equation

$$F = \frac{kmm'}{d^2}.$$

where k is a constant depending on the units employed.

39. **Weight.**—Owing to the great distance of the Earth's centre, the forces which it exerts on the particles of which bodies are composed are so nearly parallel, that they may be regarded as being actually so, and these forces may be combined into a single force called *the Weight* of the body. This single force passes *in all positions* of the body through a certain point called **The Centre of Gravity of the Body.** In mechanical problems we may consider the whole Weight of a body as a force concentrated at this point. The power of the Earth to attract bodies is called *Gravity*.

40. **Vertical and Horizontal.**—The direction in which a particle would fall freely in any place under the influence of gravity is termed a *Vertical Line:* any line or plane $\perp$ to this line is termed *Horizontal.*

The Horizon is a horizontal plane at any point of the Earth's surface.

41. 2°. **Pressure.**—When a body is *pushed* by the hand or a rod, the force so exerted is called *Pressure.*

There is considerable ambiguity attaching to the use of the word pressure. It is most appropriately used to signify *force per area*, as, for instance, when we speak of the pressure of the atmosphere as being so many lbs. per square

inch; or again, of the pressure of the wind, as being so many lbs. to the square foot. In cases like these, the total force on a given surface = pressure (*i. e.* force per unit area) × number of units of area in the surface. The word pressure in this Treatise is sometimes used to signify *any force expressed in gravitation units.*

42. 3°. **Reactions. Action and Reaction are equal, and take place in opposite directions.**— This is Newton's 3rd Law of Motion, which he thus illustrates:—

(1). If any person press a stone with his finger, his finger is pressed by the stone with an equal force in the opposite direction.

(2). If a horse draw a body by means of a rope, the horse is drawn backwards equally towards the body.

(3). If any body impinge on another, and by its action change the motion of the other, its own motion will be changed by the same amount, and in the opposite direction (*cf.* Part II., Chap. iii., 3rd Law of Motion).

We are only concerned at present with Newton's first illustration. If a ladder rest against a vertical wall, and have its end A on the ground, there will be developed two reactions, R and R' (as indicated in Fig. 3, α),

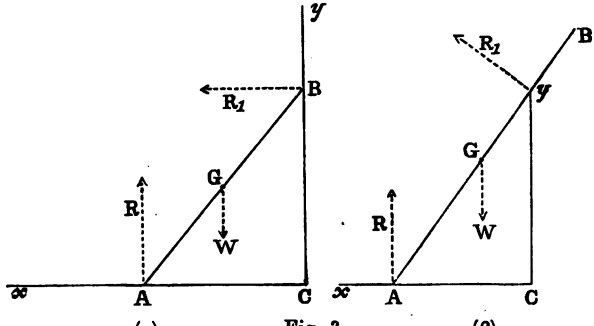


Fig. 3.

which are in fact the mechanical equivalents of the wall and ground. These reactions are not necessarily ⊥ to the wall and ground, except when these form smooth surfaces, supposed to be the case in the figures.

43. **Smooth Surfaces** are such that they exert reactions on bodies pressing against them ⊥ to the *surface of contact*.

For example, if *Cx* and *Cy* in Fig. 3 (*a*) are smooth, *R* and *R'* are respectively ⊥ thereto.

Again, if ladder overtop the wall (Fig. 3 (*β*)), *R'* is now ⊥ to ladder, *not to wall Cy*, the ladder now forming surface of contact.

EXERCISES.

1. A smooth hemispherical bowl is fixed with its rim horizontal. A beam is placed so as to overtop and lean on the rim, while one end rests on the inside surface of the bowl. Draw a diagram exhibiting the reactions generated.

2. A beam, having one end fixed by a hinge, on which it can turn, is placed so that its other *end* rests on a smooth wheel. Show that the reaction generated passes through the centre of the wheel.

44. 4°. **Tension** is simply force exerted by means of a string, or a rod used as a string (*i. e.* a perfectly flexible and inextensible body).

45. **Strings.**

(*a*) **Flexible and Inflexible.**—A perfectly flexible string is one which offers no resistance to bending. It may be bent at any point without effort. Imperfectly flexible strings do offer such resistance, according to the degree of their rigidity.

(*b*) **Extensible and Inextensible.** — Extensible strings suffer elongation under the action of forces. Inextensible do not.

String in this Treatise means perfectly flexible inextensible string, unless otherwise stated.

46. **Principle of Cord Action.**—The tension which a weightless string experiences from any cause is the same throughout its length.

The same is true if the string passes over a smooth surface (such as a peg or pulley), but not if it pass over a rough surface. If any point of a string be attached *by a knot* to another string in a state of tension, or to a weight, the branches at each side of this knot must be considered as separate strings not necessarily under same tension.

Note to Chapter II.

Abstractions of Rational Mechanics.—Rigid Bodies, Smooth Surfaces, Weightless Bars and Strings, Flexible and Inextensible Cords, are mere abstractions, and have no actual existence in Nature, so far as our experience and observation reaches. They are convenient appliances, by which we are enabled to construct a scheme of Ideal Mechanics. When we come to Nature we must allow for Elasticity, Roughness, Imperfect Flexibility, and other modifying causes.

QUESTIONS ON CHAPTER II.

1. Represent a force of 5 lbs. acting at the middle point of the base of a triangle, and passing through the vertex.

2. What is meant by the transmissibility of force?

3. A boat is attached by a rope to the shore of a lake, on which it floats freely; a person in the boat pulls the rope. What will happen, and why?

4. A horse pulls a cart along a road. According to Newton, the horse is pulled back by a force exactly equal to that which he exerts on the cart. Why, then, does the cart move?

5. A mass which weighs 81 tons at the earth's surface is taken to a distance of 6000 miles from the earth's centre. Assuming the earth's radius to be 4000 miles, find the force acting on the mass. *Ans.* 36 tons.

6. What is meant by an inflexible inextensible string?

7. State the principle of cord action.

8. What do you mean by the abstractions of Rational Mechanics?

9. Wind, having a velocity of 50 miles per hour, exerts *a pressure* of 12 lbs. per square foot. Find the total force exerted on a bridge whose surface, exposed to the direct action of the wind, is 1000 sq. ft. *Ans.* 12000 lbs.

10. Given that the density of water is 62·5 lbs. per cubic foot; find *the pressure* at a depth of 100 feet beneath the surface of a lake due to the superincumbent water? *Ans.* 6250 lbs. per square foot.

11. If in Ex. 9 the wind blow obliquely on the bridge, so as to make an angle of 60° with its exposed surface; find the total force which the bridge experiences. *Ans.* 6000 lbs.

Ex. 11 may be omitted until Art. 51 is read.

CHAPTER III.

PARALLELOGRAM OF FORCES.

47. **Parallelogram of Forces.**—*If two forces acting at a point be represented in magnitude, line of action, and direction, by two straight lines, their resultant will be represented in magnitude, line of action, and direction, by that diagonal of the parallelogram (determined by the lines) which passes through the point.**

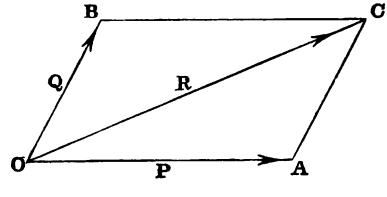


Fig. 4.

In adjoining figure, if OA and OB be the lines representing the forces, then OC represents their resultant.

48. **Experimental Proof of Parallelogram of Forces.**

Let three perfectly flexible strings be knotted together at O, and let three weights, P, Q, and R (the sum of any two of which is > third), be attached, one to the end of each cord. Pass two of the cords over smooth pegs, X and Y, and let R hang freely.

When the system has come to rest, mark off on OX and OY two lengths, OA and OB, proportional to the weights P and Q. Complete the parallelogram OC. If OC be now measured, it will be found to be proportional to R on the same scale that OA and OB were measured. Moreover, the line OC will be found to be vertical (*i.e.* in the same right line as R).

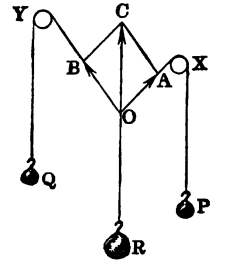


Fig. 5.

* An experimental proof of this proposition is given in the next Article; other proofs in App. I., Part II.

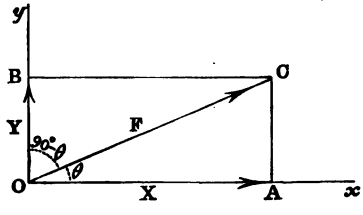
Now the material point O is in equilibrium, under the joint influence of P, Q, and R. Hence the resultant of P and Q is balanced by R. Hence R is equal in magnitade but opposite in direction to resultant of P and Q. But OC is a line of opposite direction to R, and its magnitude represents R; therefore OC represents the resultant of P and Q, both in magnitude and direction.

For other proofs of Parallelogram of Forces, see Appendix I. Part II.

49. **Resolution of Forces.**—Since P and Q (Fig. 4) give rise to R, it follows that R may be replaced by P and Q. When this is done, R is said to be resolved into P and Q, which are termed components of R (*cf.* Arts. 30, 31).

Cor.—Since (Fig. 4) OC may be the diagonal of an infinite number of parallelograms, R may have an infinite number of pairs of components.

50. **Resultant of Two Forces acting on a Point at Right Angles to each other.**



Fig, 6.

Let OA (X) and OB (Y) be the two forces, and F their resultant; then

$$F^2 = OA^2 + AC^2; \quad \text{(Euc. I. 47.)}$$

$$\therefore \quad F^2 = X^2 + Y^2;$$

$$\therefore \quad F = \sqrt{X^2 + Y^2}.$$

EXERCISES.

1. Find the resultants of the subjoined pairs of forces acting at right angles to each other on a particle—

(α) 3 and 4; (β) 5 and 12; (γ) 40 and 9.

Ans. 5, 13, 41.

2. The resultant of two forces acting on a point at right angles to each other = 61. One of the forces = 11; find the other? *Ans.* 60.

3. Two forces X and Y acting on a particle at right angles to each other give a resultant = $2X$; find the angles it makes with Ox and Oy, respectively (Fig. 6). *Ans.* $COx = 60°$. $COy = 30°$.

51. Total Effect of a Force (F) along a Line = $F \cos \theta$, where θ = angle that F makes with Line.

It is found by experiment that X and Y (Fig. 6) are totally independent of each other; *i. e.* it is found that *a force has no effect in a direction at right angles to itself.*

Hence X = whole effect of F along Ox;

but $X = F \cos \theta$; $\therefore$ &c.

X is called the resolved part of F along Ox. Hence we have the following important rule:—

To resolve a force in any direction, multiply the force by the cosine of the angle it makes with that direction.

EXERCISES.

1. Find the total effect of a force of 10 tons along a plane to which it is inclined at an angle of 60°.

Ans. Total effect = $10 \times \cos 60° = 10 \times \frac{1}{2} = 5$ tons.

2. Find resolved part of 100 lbs. along a line to which it is inclined at an angle of 45°. *Ans.* 70·71 lbs.

3. The total effect of $20\sqrt{2}$ lbs. along a plane is 20 tons. Find inclination of force to plane. *Ans.* 45°.

4. A force is inclined to the horizon at 30°. Its total effect along ground = $1\frac{1}{2}$ tons. Find magnitude of force. *Ans.* $\sqrt{3}$ tons.

5. Two forces, P and Q, are found to produce exactly the same effect along a horizontal plane, P being inclined to plane at 60°, and Q at 45°. Find ratio of P to Q.

Ans. $\frac{P}{Q} = \frac{\sqrt{2}}{1}$.

52. **Proof of the Formula** $R^2 = P^2 + Q^2 + 2 . P . Q . \cos \phi$.

Where P and Q are two intersecting forces, R their resultant, and ϕ the angle between them,

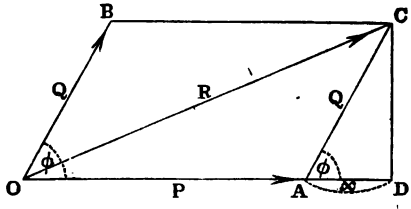


Fig. 7.

CD is $\perp$ to OA produced.

$R^2 = OA^2 + AC^2 + 2OA . x$ (Euc. II. 12);

but $x = Q \cos \phi$;

$\therefore \quad R^2 = P^2 + Q^2 + 2P . Q . \cos \phi$.

This formula is also true when the angle ϕ is either acute or right. The proofs are left to the student.

EXERCISES.

1. If $P = 6$, $Q = 7$, and $\phi = 60°$, find R. *Ans.* $R = \sqrt{127}$.

2. Find the resultant of two forces 1 and 2 lb. inclined at 120°, and show that it is $\perp$ to force 1. *Ans.* $R = \sqrt{3}$.

3. $P = 7$ lbs., $Q = 33$ lbs., $\phi = 60°$; find resultant. *Ans.* 37 lbs.

4. Two forces represented by 4 and 6 are inclined to each other at an angle of 45°; find their resultant. *Ans.* 9·26.

5. Two forces, whose magnitudes are in the ratio of 3 : 4, act at right angles to each other on a particle, and produce a resultant = 15; find forces. *Ans.* 9 and 12.

6. Required the resultant of two forces, each = Plbs., acting on a particle at an angle of 120°. *Ans.* Plbs.

N.B.—The student should note this Exercise.

7. Two forces, one of which is 2 lbs., act on a particle, and produce a resultant = 7 lbs. If the angle between the forces be 30°, find the other force. *Ans.* 5·196 lbs.

8. Round 3 smooth pegs forming an equilateral triangle an elastic ring is stretched. If the tension of the elastic band be 2 lbs., what pressure does each peg experience?

Ans. $2\sqrt{3}$ lbs.

9. If there be 6 smooth pegs forming a regular hexagon, show that the pressure which each peg experiences from an elastic band stretched round the pegs is equal to the tension on the band itself.

10. Show that, neglecting the weight of the rope, it is easier for a horse to tow a boat along a canal with a long than with a short rope.

11. What is meant by the words component, resultant, balancing force? A force of $16\frac{2}{3}$ lbs. can be balanced by two other forces of 10 and $13\frac{1}{3}$ lbs. What is the inclination of the last two forces to each other?—(*The Previous Cambridge*).

Ans. 90°.

12. Having given the magnitudes of two component forces, and of their resultant, demonstrate a formula for the resultant. From this formula, prove that the resultant cannot be less than the difference between the components. If the components be 99 lbs. and 20 lbs., acting at right angles, find the resultant.—(*The Previous*).

Ans. 101 lbs.

13. Show that three equal coplanar forces, mutually inclined at angles of 120°, and all acting inwards or all outwards, are in equilibrium.

14. Forces of 10, 11, and 12 lbs. act outwards from a point along coplanar right lines mutually inclined at angles of 120°. Find the magnitude of their resultant, and its inclination to the force 11.

Ans. $\sqrt{3}$; 90°.

15. If the right lines OA and OB represent forces, prove that their resultant $= 2 \cdot OM$ where M is the middle point of AB.

N.B.—This useful theorem should be remembered.

16. Two chords OA and OB of a circle represent two forces; if OA be fixed, find the position of OB, when the resultant of OA and OB is a maximum.

Ans. Describe a circle touching the given circle at A, and passing through its centre. This circle is the locus of the middle points of the chords AB. If X be the centre of this touching circle, OX is the direction of the maximum resultant. If OD be the diameter of original circle through O, and if $AOD = \alpha$ and $BOD = \theta$,

$$\cot(\alpha - 2\theta) = 3 \cot \alpha.$$

17. If the base AB of a triangle ABC be so divided at D that $AD/DB = n/m$, and if CD be joined; prove that if forces represented by mCA and nCB act on a particle at C, their resultant will be represented by $(m+n)CD$.

18. Three forces acting on a particle keep it at rest; they are proportional to $\sqrt{3}+1$, $\sqrt{6}$, and 2. Find the angles at which they are inclined to each other. *Ans.* 105°, 120°, 135°.

53. Resultant of Three or more Forces acting on a Particle.

Find resultant of any two; compound this with 3rd, and so on.

For another method, compare Art. 62.

EXERCISES.

1. If OA, OB, and OC represent three forces not coplanar, find their resultant.

Ans. OD, where OD is the diagonal of the parallelopiped, determined by OA, OB, and OC.

2. Show that resultant of X, Y, and Z mutually at right angles

$$= \sqrt{X^2 + Y^2 + Z^2}.$$

3. If R = resultant in 2; prove that

$$X = R \cos \alpha,$$

$$Y = R \cos \beta,$$

$$Z = R \cos \gamma;$$

where α, β, and γ are angles made by R with X, Y, and Z, respectively.

CHAPTER IV.

THE TRIANGLE OF FORCES.

54. **Important Lemma.**

If two **Concurrent Forces** (*i. e.* forces acting on a particle) be represented in magnitude and direction by the sides of a plane triangle *taken in order*,* then will the third side *taken in reverse order* represent their resultant. (Fig. 8, p. 21.)

For OA and OB give OC; therefore forces represented by OA and AC, *acting on a particle*, give OC; therefore, &c.

55. **Triangle of Forces.**—If three forces, represented in magnitude and direction by the sides of a plane triangle *taken in order*, act on a particle, they will equilibrate. (Fig. 8, p. 21.)

For forces represented by OA and AC on a particle give OC; therefore OA, AC, and CO, give OC and CO, which equilibrate.

56. **Converse of Triangle of Forces.**†—Three coplanar forces, acting on a particle, which equilibrate, may be represented in magnitude and direction by the sides of a plane triangle *taken in order*.

---

* **Taken in Order.**—The sides of any rectilinear figure are said to be *taken in order* when taken as they would be traversed by a point moving continuously round the figure, with either watchwise or contra-watchwise rotation (Art. 65). Thus the sides of the quadrilateral $ABCD$, *taken in order*, are either AB, BC, CD, DA, or AD, DC, CB, BA, which latter is the *reverse order* of the former.

† This is not the strict logical converse, and therefore requires proof.

For R is equal and opposite to OC; therefore $P : Q : R = OA : AC : CO$.

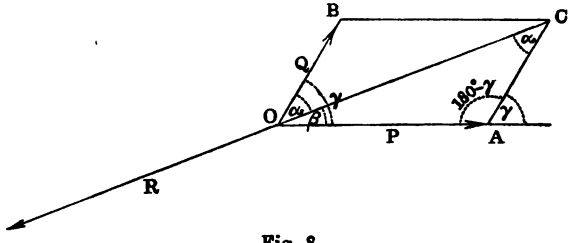


Fig. 8.

Cor.—The coplanar condition need not be specified; for R being equal and opposite to OC, *must* be coplanar with P and Q.

57. Conditions that Three Concurrent Forces should Equilibrate.

They must be—

1°. Coplanar.

2°. Proportional to sides of a plane triangle *taken in order*.

For they can be represented by sides of OAC *taken in order*. (Fig. 8.)

58. Conditions that any Three Non-parallel Forces P, Q, and R, acting on a Rigid Body, should Equilibrate.*

They must be—

1°. Coplanar.

2°. Concurrent.

3°. Proportional to sides of a plane triangle *taken in order*.

1°. To show that three forces P, Q, and R, which act on a rigid body and equilibrate, are coplanar.

* This Art. may be omitted till Chapter V. is read.

The equilibrium of the body is not disturbed by fixing any two points in it. Fix

x on P, which neutralises it;

y on Q, ,, ,, ;

then R must intersect the straight line xy. For otherwise the body would rotate about it (Arts. 63, 64). Similarly, R must intersect xy', where y' is on Q; therefore, since x is *any* point on Q, P, Q, and R are coplanar.

2°. To show that P, Q, and R meet in a point. Since P and Q are coplanar, they must meet if produced. To secure equilibrium, R must be equal and opposite to their resultant, and therefore passes through intersection of P and Q.

3°. Proved as in Art. 57.

59. **If three Forces, P, Q, and R, be in Equilibrium,**

$$\frac{P}{a} = \frac{Q}{b} = \frac{R}{c};$$

where a, b, and c are the sides of any triangle whose sides are parallel, perpendicular, or isoclined to P, Q, and R, respectively. (Fig. 8, p. 21.)

For $$P : Q : R = OA : AC : CO.$$

Now any triangle whose sides are parallel, perpendicular, or isoclined to P, Q, and R, is similar to OAC;

$$\therefore \quad OA : AC : CO = a : b : c;$$

$$\therefore \quad P : Q : R = a : b : c;$$

$$\therefore \quad \frac{P}{a} = \frac{Q}{b} = \frac{R}{c}.$$

Example.

A weight of 100 lbs. is sustained by two cords, CA and CB, attached to two points, A and B, in a horizontal line, and 5 feet apart. If CA and CB be 3 and 4 feet respectively in length, determine the tensions on CA and CB.

$3^2 + 4^2 = 5^2$; $\therefore\ b^2 + a^2 = c^2$, where a, b, c are sides opposite A, B, and C, respectively; $\therefore\ C = 90°$.

Also c is perpendicular to W; $\therefore$ triangle ABC has its sides perpendicular to forces;

$$\therefore\quad \frac{T}{4} = \frac{W}{5} = \frac{T_1}{3};$$

whence

$$T = 80 \text{ lbs.}, \text{ and } T_1 = 60 \text{ lbs.}$$

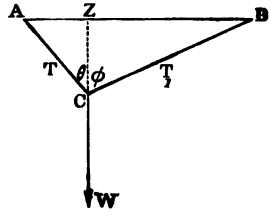


Fig. 9.

EXERCISES.

1. If $AB = 41$, $BC = 40$, and $CA = 9$, find T and T_1 when $W = 82$ tons.
   Ans. $T = 80$ tons; $T_1 = 18$ tons.
2. If $AB = 61$, $BC = 60$, $CA = 11$, find T and T_1 when $W = 244$ lbs.
   Ans. $T = 240$ lbs.; $T_1 = 44$ lbs.
3. If $AB = BC = CA = a$, and $W = 100$ lbs., find T and T_1.
   Ans. $T = T_1 = 57{\cdot}7$ lbs.
4. If $AB = a\sqrt{2}$, $BC = CA = a$, find T and T_1 when $W = 20$ tons.
   Ans. $T = T_1 = 10\sqrt{2}$ tons.
5. If $AB = 2a$, $BC = CA = l$; show that $T = T_1 = \dfrac{Wl}{2\sqrt{l^2 - a^2}}$.

60. **If three Forces, P, Q, and R, acting on a Particle, are in Equilibrium** (Fig. 8, p. 21),

$$\frac{P}{\sin \overset{\wedge}{QR}} = \frac{Q}{\sin \overset{\wedge}{RP}} = \frac{R}{\sin \overset{\wedge}{PQ}},$$

where $\overset{\wedge}{QR}$ means the angle between Q and R.

Proof.— $P : Q : R = OA : OB : OC$ (OC being $= R$)

$= OA : AC : CO$

$= \sin \alpha : \sin \beta : \sin (180° - \gamma)$

$= \sin \alpha : \sin \beta : \sin \gamma$

$= \sin \overset{\wedge}{QR} : \sin RP : \sin \overset{\wedge}{PQ}$;

$$\therefore\quad \frac{P}{\sin \overset{\wedge}{QR}} = \frac{Q}{\sin \overset{\wedge}{RP}} = \frac{R}{\sin \overset{\wedge}{PQ}}.$$

EXERCISES.

1. A force (R) of 100 lbs. is in equilibrium with forces P and Q. If $\hat{RP} = 30°$, and $\hat{QR} = 60°$, find P and Q.

Here $\hat{PQ} = 90°$.

Then $$\frac{P}{\sin 60°} = \frac{100}{\sin 90°};$$

whence $$P = 100 \sin 60° = \frac{100\sqrt{3}}{2} = \frac{100 \times 1{\cdot}732}{2} = \frac{173{\cdot}2}{2} = 86{\cdot}6 \text{ lbs.}$$

Similarly, $Q = 50$ lbs.

2. A force of 10 lbs. is in equilibrium with two forces P and Q, being inclined to P at an angle of 90°, and to Q at an angle of 30°; find P and Q.

Ans. $P = \frac{10}{\sqrt{3}}$ lbs.; $Q = \frac{20}{\sqrt{3}}$ lbs.

3. A weight of 20 lbs. hangs from a fixed point by means of a flexible cord. A horizontal force is applied to the weight, so as to deflect it from the vertical line through an angle of 45°. Determine the magnitude of the deflecting force, and the tension on the string.

Ans. Deflecting force = 20 lbs.
Tension on string = $20\sqrt{2}$ lbs.

4. A smooth ring sustaining a weight W is strung on a flexible cord whose length is $2c$ inches. The ends of the string are attached to two fixed points in a horizontal line $2a$ inches apart. Find the tension on the string.

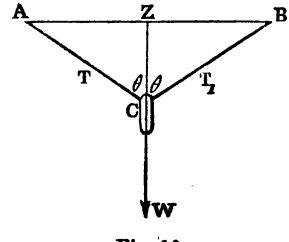


Fig. 10.

Since the ring is smooth, it will occupy middle position; whence

$$T = \frac{Wc}{2\sqrt{c^2 - a^2}}. \quad \textit{Ans.}$$

5. Two cords, having equal weights (P, P) attached to their ends, pass over smooth pegs (A and B) in the same horizontal line, and are joined to a third weight (W) at C. Determine the magnitude of W, such that the vertical distance of C from AB may be b feet, the distance AB being $2c$ feet.

$$Ans.\ W = \frac{2Pb}{\sqrt{c^2 + b^2}}.$$

61. **Resultant (R) of several Concurrent Forces represented** (*cf.* Art. 53).*

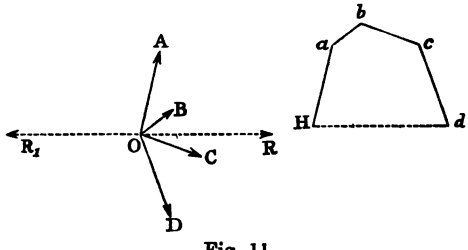


Fig. 11.

Take any point H.

Draw Ha = parallel, and codirectional to OA.
ab ,, ,, ,, OB.
bc ,, ,, ,, OC.
cd ,, ,, ,, OD;

then Hd *represents* resultant.

For Ha and ab give Hb. (Art. 54)
Hb and bc ,, Hc.
Hc and cd ,, Hd;
$\therefore$ R is *represented* by Hd.

---

* The triangle and polygon of forces are the foundation of **Graphic Statics**, in which forces are calculated by measuring the lengths of lines.

62. **Polygon of Forces.**—If a system of concurrent forces can be represented in magnitude and direction by the sides of a closed polygon *taken in order*, the system equilibrates.

For (Fig. 11), when d coincides with H,

$$Hd = 0; \text{ and } \therefore R = 0.$$

EXERCISES.

1. $OBCA$ is a parallelogram; BC, CO, and AB represent concurrent forces; show that BC represents a force which equilibrates them.

2. AB and CD are intersecting right lines; show that DC represents a force which equilibrates forces represented by AB, BD, and CA.

3. Show that concurrent forces represented by OA, OB, OC, OD, and OR_1, equilibrate (Fig. 11).

Important Remark.—The student must be careful to observe that the forces in the Polygon of Forces do not act along its sides, *but at a point.* The polygon (or triangle) is merely an auxiliary figure to *represent* magnitude and direction of resultant, *but not point of application.*

CHAPTER V.

MOMENTS.

63. **Definition of a Moment.**—The moment of a force *about a point* is the product of the force and the perpendicular on its direction from the point.

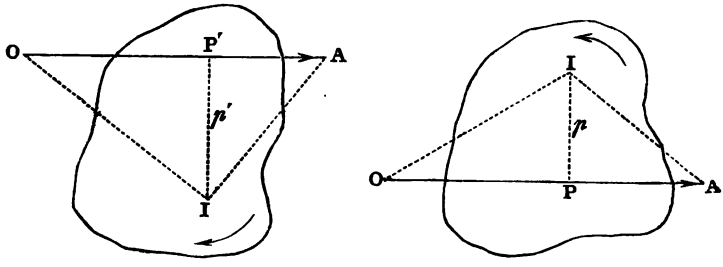


Fig. 12.

If OA be the force (P), and I the point; then dropping p perpendicular on P from I, moment of P about $I = Pp$.

The moment of a force (P) *about a line* (L) is thus obtained—
Resolve P into two components—

Y parallel to L,

X perpendicular to L.

Let p = perpendicular from L on X; then

Moment of P about $L = Xp$.

The moment of a force (P) *about a plane* $= Pp$, where p is the perpendicular distance of P's point of application from plane.

64. **Physical Meaning of a Moment.**—The moment of a force about a point is the measure of the force's tendency to turn a body on which it acts round the point.

For example, consider the door of a sitting-room which revolves on an axis through its hinges. If a force of, say 3 lbs., be applied at a distance of 2 feet from this axis perpendicular to the door, the amount of tendency to turn the door on its hinges may be represented by $3 \times 2 = 6$. Again, if the same force be applied at a distance of $\frac{2}{3}$ foot from the axis, the value of the moment *is* $3 \times \frac{2}{3} = 2$; so that the moment in this case is only $\frac{1}{3}$ of that in the former case.*

65. **Signs of Moments.**—We adopt the following convention :—Moments are + or −, according as they produce watch-wise (same as watch-hands, Fig. 12 α), or contra-watchwise rotations (Fig. 12 β).

66. **Rule to determine the Sign of a Moment.**—Conceive a person proceeding along the force arrowwise (towards arrow head). Then, moment is + or −, according as the point about which the moments are taken is to his right or left-hand side. (Fig. 12.)

67. **Graphic Representation of the Moment of a Force about a Point.**—

Since (Fig. 12) $Pp = 2\Delta \,.\, OIA$;

therefore moment of P about $I = 2\Delta$, whose base is the force, and vertex the point about which moments are taken.

68. **The Moment of a Force about a Point = Parallelogram** *OL*.

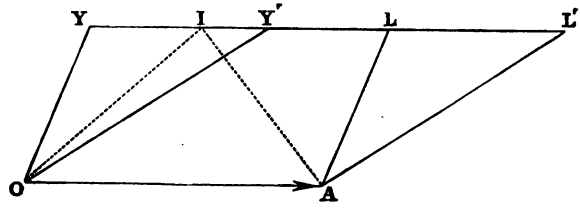


Fig. 13.

For moment of OA about $I = 2OIA = \square\, OL = \square\, OL'$.

Hence—*The moment of a force about a point = any parallelogram having force for one side, and opposite side passing through point.*

* Momental effects can be expressed in units of work : *cf.* Part II. Chap. v.

69. **The Moments of two Concurrent Forces about any Point on their Resultant are equal and opposite.—**

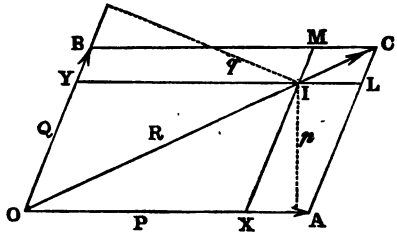


Fig. 14.

Let I = point on resultant.

Moment of P about $I = \square\, OL$.

„ Q „ $I = \square\, OM$.

Now $AI = BI$ (Euc. I. 43);

$OI = OI$.

Add, and $OL = OM$;

i. e. moment of P = moment of Q (about I);

$\therefore\ Pp = Qq$.

The moments, therefore, are equal in magnitude, and they evidently differ in sign (Art. 66).

70. **If the Moments of two Forces (P and Q) about any Point (I) in their Plane be equal and opposite, the Point lies on their Resultant.**

For if not, let point be I';

then $OL' = OM$ (hyp.);

but $OL = OM$ (Art. 69);

$\therefore\ OL' = OL$, which is absurd;

$\therefore$ &c.

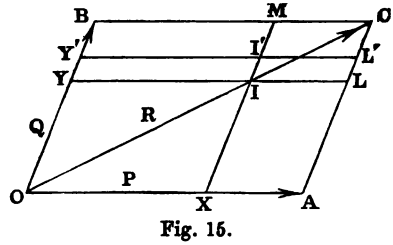


Fig. 15.

Example (Woolwich, 1885).

Find the least horizontal force applied at the centre of a wheel which will cause it to surmount a given obstacle, given the vertical load acting at the centre of the wheel, and supposing the wheel to have no velocity when in contact with the obstacle.

Reaction (R) of obstacle (h) is in equilibrium with F and W, and therefore passes through O (their intersection); therefore resultant of F and W passes through C;

$$\therefore \quad Ff = Ww \quad \text{(Art. 69)};$$

$$\therefore \quad F(r-h) = W\sqrt{h(2r-h)};$$

$$\therefore \quad F = W\frac{\sqrt{h(2r-h)}}{r-h}.$$

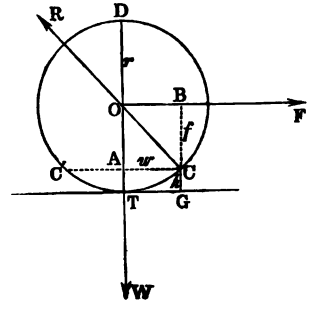


Fig. 16.

EXERCISES.

1. A weight of 100 lbs. is suspended from a fixed point P by a flexible string. What horizontal force applied to the string will draw the upper portion aside through an angle of 30°?

Ans. Taking moments round P, force = 57·7 lbs.

2. In Ex. 1, if force = 50 lbs., find through what $\angle$ it deflects the string.

Ans. If $\theta = \angle$, $\tan\theta = \frac{1}{2}$.

3. A heavy uniform beam rests with one end (A) on a smooth horizontal plane (CA), and the other (B) against a smooth vertical wall (CB), and is sustained in this position by a cord attached to the end (A) of the beam, and to the intersection (C) of the plane and wall; determine the tension on the cord.

Ans. $T = \dfrac{W\cot\theta}{2}$, where θ = inclination of beam to horizon.

N.B.—Take moments about intersection of the reactions of plane and wall, and thereby get rid of unknown reactions. This method should be carefully observed.

71. **The Algebraic Sum of the Moments of two Concurrent Forces about any Point in their Plane = Moment of their Resultant about same Point.**

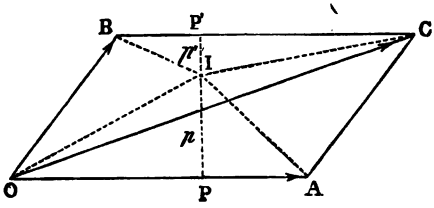


Fig. 17.

For I being the point (Fig. 17),

$$\Delta\, OIA = \tfrac{1}{2}\, OA \times p'\,;\quad \Delta\, BIC = \tfrac{1}{2}\, BC \times p' = \tfrac{1}{2}\, OA \times p'\,;$$

$$\therefore\quad \Delta\, OIA + \Delta\, BIC = \tfrac{1}{2}\, OA\,(p + p') = \tfrac{1}{2}\, OA \times PP' = \tfrac{1}{2} \square\, OC.$$

Also
$$\Delta\, OIB + \Delta\, BIC + \Delta\, OIC = \tfrac{1}{2} \square\, OC\,;$$

$$\therefore\quad OIA + BIC = OIB + BIC + OIC\,;$$

$$\therefore\quad OIA = OIB + OIC\,;$$

$$\therefore\quad OIB - OIA = -\, OIC\,;$$

$$\therefore\quad 2OIB - 2OIA = -\, 2OIC\,;$$

therefore moment of OB + moment of OA = moment of OC, the moments of OA and OC in fig. being negative.

A similar proof holds, no matter where I be situated.

72. **If three Concurrent Forces be in Equilibrium, the Sum of their Moments round any Point (I) in their Plane = 0.**

For (Fig. 17), let $OC' = OC$ reversed; then

Moment of $OC' = -$ moment of OC;

but moment of OA + moment of OB = moment of OC (Art. 71);

$\therefore$ moment of OA + moment of OB + moment of $OC' = 0$.

Now OC' = resultant (R) of OA (P) and OB (Q);

$\therefore$ moment of P + moment of Q + moment of $R = 0$,

or
$$Pp + Qq + Rr = 0,$$

where p, q, and r are perpendiculars on P, Q, and R, respectively, from I.

Cor.—If any number of concurrent coplanar forces equilibrate, the sum of their moments round any point in their plane = 0,

or
$$\Sigma(Pp) = 0.$$

CHAPTER VI.

PARALLEL FORCES.

73. Magnitude, Direction, and Point of Application, of Resultant (R) of two like Parallel Forces (P and Q) acting on a Rigid Body.

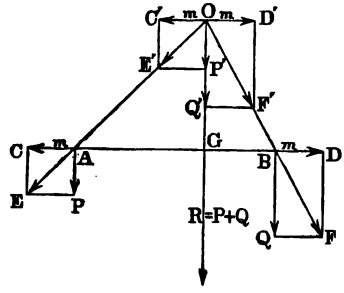


Fig. 18.

In the directions of P and Q take any two points A and B, and introduce at A and B two equal and opposite forces (m, m) acting in the line AB. These forces, being in equilibrium, will not affect the action of P and Q.

Resultant of P and m is AE.

Resultant of Q and m is BF.

Remove the points of application of these resultants to O, their point of intersection, and resolve them there into their original components. We then have acting at O two equal and opposite forces (m, m), which neutralize each other, and two *like* forces P and Q acting along OG;

$$\therefore \quad R = P + Q. \qquad (1)$$

Again, $\Delta^s APE$ and OGA are similar;

$$\therefore \quad \frac{m}{P} = \frac{AG}{OG},$$

$$\therefore \quad P \times AG = m \times OG.$$

Similarly, $Q \times BG = m \times OG$ (from $\Delta^s BQF$ and OGB);

$$\therefore \quad P \times AG = Q \times BG. \qquad (2)$$

The results of this Article may be thus stated:—

The resultant of two like parallel forces is parallel to the forces, equals their sum, and divides the distance AB between their points of application, so that each force multiplied by the segment of AB adjacent to it has the same numerical value.

74. Magnitude, Direction, and Point of Application, of Resultant (R) of two unlike Parallel Forces (P and Q) acting on a Rigid Body.

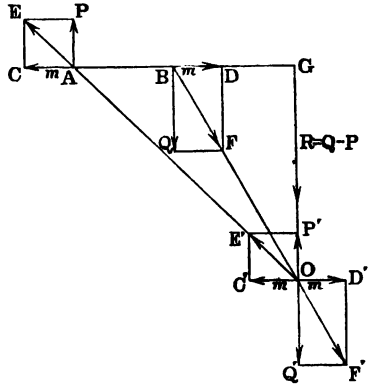


Fig. 19.

The same construction being adopted as in Art. 73, we have finally two equal and opposite forces (m, m) acting at O which destroy each other, and two *unlike* parallel forces P and Q acting along GO;

$$\therefore \quad R = Q - P. \qquad (1)$$

Again, $\Delta^s APE$ and OGA are similar;

$$\therefore \quad \frac{m}{P} = \frac{AG}{OG};$$

$$\therefore \quad P \times AG = m \times OG.$$

Similarly, $Q \times BG = m \times OG$ (from $\Delta^s BQF$ and OGB);

$$\therefore \quad P \times AG = Q \times BG. \qquad (2)$$

We thus have the following results:—

The resultant of two unlike parallel forces is parallel to the forces, equals their difference, and divides the distance AB between their points of application externally on the side of the greater force, so that each force multiplied by the segment of AB adjacent to it has the same numerical value.*

EXERCISES.

1. If $AB = 14$ inches, $P = 2$ lbs., and $Q = 5$ lbs., find the position and magnitude of R—

(α) When P and Q are like parallel forces;

(β) ,, ,, unlike ,,

(α) Here $AG + BG = 14$,

$2AG = 5BG$;

whence $AG = 10$, $BG = 4$.

Also, $R = 7$ lbs.

(β) Here $AG - BG = 14$,

$2AG = 5BG$;

whence $AG = 23\frac{1}{3}$, $BG = 9\frac{1}{3}$.

Also, $R = 3$ lbs.

2. The resultant of two like parallel forces = 20 lbs., and acts at a point three times as near to P as to Q; find P and Q.

---

* The segments of a right line, made by a point G on its direction, are the two distances GA and GB of the point from the extremities of the line, whether that point be on AB (Fig. 18), or on AB produced (Fig. 19). In this latter case GB must be considered negative.

Let $AG = x$; then $BG = 3x$.

Then $P \times x = Q \times 3x$ (Art. 73); (2)

$\therefore\ P = 3Q$.

Also, $P + Q = 20$;

whence $P = 15,\ Q = 5$.

3. The resultant of two parallel unlike forces is 2 lbs., and acts at distances 6 inches and 8 inches from them; find the forces.—(*The Previous Cambridge.*) *Ans.* 6 lbs. and 8 lbs.

4. Parallel forces of 3 lbs. and 7 lbs. act in the same direction at points A and B, distant 20 inches apart. How far from the middle of AB does their resultant cut AB?—(*The Previous Cambridge.*) *Ans.* 4 inches.

5. Parallel forces of 4 lbs. and 9 lbs. act in opposite directions at points A and B, 15 inches apart. How far from the middle of AB does their resultant cut AB produced?—(*The Previous Cambridge.*) *Ans.* $19\frac{1}{2}$ inches.

6. Find the position and magnitude of the resultant of two parallel forces P and Q acting towards the same parts.

Show that if the force P be changed to $\frac{Q^2}{P}$, the new resultant will occupy the same position as if the forces P and Q had been interchanged.—(*The Previous Cambridge.*)

7. Two parallel forces P, Q act towards opposite parts. Find the magnitude and line of action of their resultant.

Show that if Q be changed to $\frac{P^2}{Q}$, the line of action of the resultant will occupy the same position as if the forces had been interchanged.—(*The Previous Cambridge.*)

75. The Moments of two Parallel Forces (which have a Resultant)* about any Point on their Resultant are equal in Magnitude and opposite in Sign.—

For (Arts. 73, 74)—

$$P \times GA = Q \times GB.$$

Let AB be perpendicular to forces, and the proposition is proved as regards magnitude; and since the moments have opposite signs (Art. 65); therefore &c.

---

* This proviso added to exclude couples (Chap. VII.).

76. The Sum of the Moments of two like Parallel Forces about any Point in their Plane = Moment of their Resultant about that Point.—

In Fig. 18, p. 32, let D be the point, and let AD, BD, and $GD = p$, q, and r, respectively; then

$$Pp + Qq = Rr.$$

For $P \times AG = Q \times BG.$

Now $AG = AD - GD = p - r,$

and $BG = GD - BD = r - q;$

$$\therefore \quad P(p - r) = Q(r - q);$$

whence $Pp + Qq = (P + Q)r,$

or $Pp + Qq = Rr \ (R = P + Q).$

77. The Difference of the Moments of two unlike Parallel Forces (which have a Resultant) about any Point in their Plane = Moment of their Resultant about that Point.—

In Fig. 19, p. 33, let D be the point; and let AD, BD, and $GD = p$, q, and r, respectively; then

$$P \times AG = Q \times BG;$$

$$\therefore \quad P(p + r) = Q(q + r);$$

$$\therefore \quad Pp - Qq = (Q - P)r = Rr.$$

Cor. 1.—It follows from (76 and 77) that

The algebraic sum of the moments of any two parallel forces (which have a resultant) about any point in their plane = moment of their resultant about the same point.

Cor. 2.—The sum of the moments of a set of coplanar parallel forces (which have a resultant) about any point in the plane of the forces = moment of their resultant about the same point.

This principle may be thus expressed—

$$(P_1x_1 + P_2x_2 + P_3x_3 + \&c.) = (P_1 + P_2 + P_3 + \&c.)\,\bar{x},$$

or $$\Sigma(Px) = \bar{x}\Sigma(P);$$

where P_1, P_2, P_3, &c., are the forces;

x_1, x_2, x_3, &c., their distances from point;

$\bar{x}$, distance of resultant from point,

and Σ a symbol of summation.

EXERCISES.

N.B.—In the following Exercises it is assumed that the weight of a uniform homogeneous bar may be considered as concentrated at its middle point (*C. G*). The reason of this assumption will be understood when Chap. ix. is read. Also observe that a beam on which weights are hung will *balance* on the point of its length through which the resultant of its own and the hung weights passes.

1. A bar weighing 6 lbs., and 12 feet long, has weights of 2, 8, 4, and 16 lbs. hung along it, as indicated in the diagram; find the point on which it will balance.

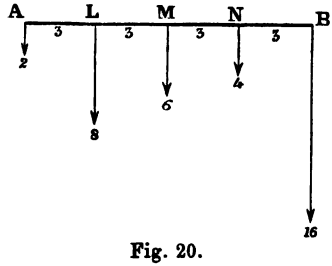


Fig. 20.

Moments round *A* gives (Art. 76)—

$$2\,(0) + 8\,(3) + 6\,(6) + 4\,(9) + 16\,(12) = (2 + 8 + 6 + 4 + 16)\,\bar{x},$$

whence $$\bar{x} = 8;$$

therefore bar will balance at a point 8 feet from *A*.

2. Parallel forces, as indicated in the diagram, act along a line. Find the point in the line through which their resultant passes.

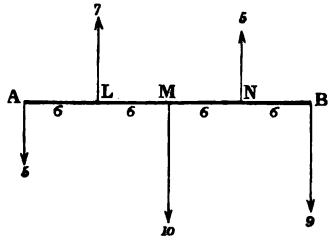


Fig. 21.

Taking moments round A gives (*Cor.* 1, Art. 77)—

$$5\,(0) - 7\,(6) + 10\,(12) - 5\,(18) + 9\,(24) = (5 - 7 + 10 - 5 + 9)\,\bar{x},$$

whence $\bar{x} = 17$ (units from A).

3. Three parallel forces, 11, 12, 13, act at points on a line distant respectively 4, 5, and 6 feet from a point (O) on the line, the first force being opposite in direction to the other two; determine the magnitude and position of the resultant.—(*Degree*, 1875, Mr. M'Cay.)

Ans. $R = 14$, and acts at a distance $= 6\frac{5}{7}$ feet from the point O.

4. A system of parallel forces equal respectively to 1, 2, 3, 4, 5, 6, and 7 lbs. act at points distant 1, 2, 3, 4, 5, 6, and 7 feet, respectively, from a fixed point, O; determine the magnitude and point of application of their resultant (R). *Ans.* $R = 28$ lbs., and acts 5 feet to right of O.

5. A system of n parallel forces, equal respectively to 1, 2, 3, ... n lbs., act at points distant respectively 1, 2, 3, ... n feet from a fixed point, O; show that the resultant $= \frac{n \cdot n+1}{2}$ lbs., and acts at a point $\frac{2n+1}{3}$ feet distant from O.

6. A uniform bar 10 feet long weighs 30 lbs., and has weights of 40 lbs. and 50 lbs. suspended from its extremities; find at what point the bar will balance. *Ans.* 5 inches from middle adjacent to 50 lbs.

7. A uniform bar 5 feet in length, having weights of 5 lbs. and 8 lbs. respectively suspended from its extremities, balances on a fulcrum 6 inches distant from its middle point; find the weight of the bar. *Ans.* 2 lbs.

8. A uniform rod 12 feet long weighs 10 lbs.; if it rest on a fulcrum 4 feet from one end, find what weight must be suspended from that end to balance 5 lbs. hanging at the other end. *Ans.* 15 lbs.

9. In Fig. 21, p. 38, if weights of 2, 4, 6 and 8 lbs. be hung at A, L, N and B, respectively, what must the weight of the beam be that it may balance about a point 2 feet to right of M? *Ans.* 22 lbs.

78. The Sum of the Moments of a System of Coplanar Parallel Forces in Equilibrium about any Point in their Plane = 0.

It might appear that if $R = 0$ in (*i. e.* $\Sigma(P) = 0$) *Cor.* 2, Art. 77, that then

$$\Sigma(Px) = 0.$$

This is, however, not the case; as even when $R = 0$, the forces may give rise to a couple (Art. 80), which has a constant moment about any point in its plane (Art. 85). If, however, there is neither a resultant force (R) nor a resultant couple (*i. e.* if the forces equilibrate), then

$$\Sigma(Px) = 0.$$

EXERCISES.

1. A beam 12 feet long, and which weighs 24 lbs., is supported on props at A and N, as shown in the diagram. If weights of 12 lbs. and

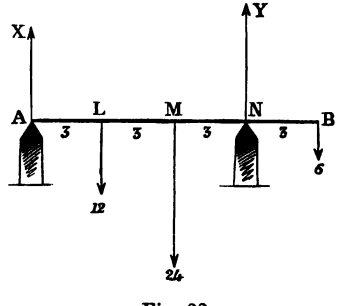


Fig. 22.

6 lbs. be hung at L and B, as shown, find the pressures on the props.

The props generate two reactions X and Y, as shown, which are in equilibrium with the weight of the beam and the appended weights;

$$\therefore \quad \Sigma(Px) = 0.$$

Taking moments about N,

$$9X - 12 \times 6 - 24 \times 3 + 6 \times 3 = 0,$$

whence $$X = 14 \text{ lbs.}$$

Similarly, taking moments about A,

$$Y = 28 \text{ lbs.}$$

2. A uniform bar, 12 feet long and 40 lbs. weight, rests in a horizontal position on two props placed at its extremities. If a weight of 112 lbs. be hung at 1 foot from one extremity, and 100 lbs. at 2 feet from the other; find the pressures on the props.—(*Degree*, *T.C.D.*, 1883.)

Ans. $139\frac{1}{3}$ lbs. and $112\frac{2}{3}$ lbs.

3. A uniform bar, 12 feet long and 112 lbs. weight, is supported on two props, respectively situated 1 foot and 4 feet from the ends of the bar; find the pressures on the props.—(*J. S.*, *Trinity*.) *Ans.* 32 lbs. and 80 lbs.

4. A beam 18 feet long, and weighing 24 lbs., has weights of 6 lbs. and 12 lbs. hung at distances from its ends 3 and 4 feet, respectively. If the beam be supported by props at its ends, determine the pressures they experience. *Ans.* $19\frac{2}{3}$ lbs. and $22\frac{1}{3}$ lbs.

5. A horizontal beam 12 feet long, and weighing 10 lbs. per foot, sustains a weight of 240 lbs. at one-third of its length from one end, and an upward thrust of 30 lbs. at one-third of its length from the other end; find the pressures on the supports which are placed at its ends.—(*J. S.*, *Michaelmas*, 1883.) *Ans.* 210 lbs. and 120 lbs.

CHAPTER VII.

COUPLES.

79. Case of two equal and unlike Parallel Forces, acting at different Points on a Rigid Body.

If the forces P and Q in Art. 74 are equal, the equation

$$P \times GA = Q \times GB \text{ gives}$$

$$GA = GB,$$

which can only be true when G is at infinity.

Also, $$R = P - P = 0.$$

We thus have a resultant = 0 acting at infinity. These anomalous results indicate a failure of the method in this case.

80. Definition of a Couple.

Two equal and unlike parallel forces, acting at different points of a rigid body, form what is called a couple.

The Arm of a couple is the perpendicular distance between its two forces.

The Moment of a couple is the product of the arm and one of the forces.

The Plane of a couple is the plane of its forces.

The Axis of a couple is a right line drawn anywhere perpendicular to the plane of the couple, its length being proportional to the moment of the couple.

The couple $P.p$ or $P.AB$ means the couple whose force $= P$ and arm $= p$ or AB.

81. Positive and Negative Couples.

Suppose the middle point of the arm of a couple to be fixed, but so as to allow rotation round it.

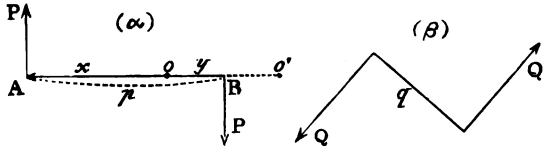


Fig. 23.

Then— Fig. (α) represents a + couple.

Fig. (β) ,, a − couple.

Watch-hand rotation being considered +.

Contra watchwise ,, ,, −.

82. Positive and Negative Face of a Couple.

The direction in which any rotation appears to take place depends on the position of the observer.

Every couple is therefore + or −, according as it is viewed from one side of its plane or from the opposite side.

The + face of a couple is the side from which it appears +, the − face that from which it appears negative.

83. Positive and Negative Axis of a Couple.

The axis of a couple is + or −, according as it points away from the + or − face of the couple.

84. Effect of a Couple.

The effect of a couple on a free rigid body is to make the body turn round. It is shown in works on rigid dynamics that this rotation takes place round some straight line, passing through a certain point in the body called its centre of gravity (Art. 94), but this line is not necessarily perpendicular to the plane of the couple.

85. **The Algebraical Sum of the Moments of two Forces forming a Couple about any Point in the Plane of the Couple is Constant, and equal to the Moment of the Couple.**

Draw a common perpendicular to the forces through the point O (Fig. 23a); then the moments of the forces are both + (Art. 66); therefore their sum $= P \cdot x + Py = P(x + y) = Pp$ = moment of couple.

If O' (Fig. 23a) be the point; then sum of moments $= P(OA - OB) = P \cdot p =$ moment of couple.

86. **Two Coplanar Couples (Pp and Qq), of equal and opposite Moments, Equilibrate.**

We have to consider two cases—

1°. When P and Q are *not* parallel;
2°. ,, are parallel.

1°. When P and Q are *not* parallel:—

Let AA' be a parallelogram formed by lines of action of forces (PP) and (QQ).

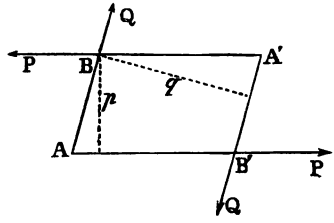


Fig. 24.

$Pp = Qq$ (by hypothesis);

$\therefore$ resultant of P and Q at B' passes through B (Art. 70);

$\therefore$,, ,, acts along BB'.

Similarly, ,, ,, B acts along $B'B$.

Latter resultant = former; and, as they act in opposite directions along the same line, they equilibrate; therefore, &c.

2°. Let the forces (PP) be parallel to (QQ):—

Introduce two new couples $(P'p')$ and $(Q'q')$ coplanar with original couples of equal moment and sign to (Pp) and (Qq), respectively, and such that P' and Q' are neither parallel to each other nor to P and Q.

By 1°, these couples are in equilibrium, and therefore their introduction does not affect the action of (Pp) and (Qq), the original couples. Now, by

1°. $P'p'$ equilibrates Qq,

and $Q'q'$,, Pp;

therefore the four couples are in equilibrium. Removing the introduced couples $(P'p')$ and $(Q'q')$, we have

(Pp) in equilibrium with (Qq),

which proves the proposition.

87. The Effect of a Couple is not altered by Transporting the Couple to any Plane parallel to original Plane of Couple, the Arm remaining parallel to its original Position.

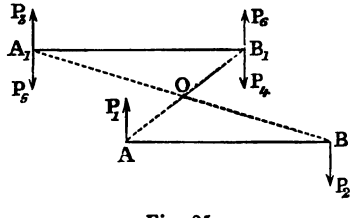


Fig. 25.

Let AB be the arm and A_1B_1 its new position, parallel to AB. Join A_1B and $A_1'B$ (bisecting each other at G). Apply at A_1' and B_1' P_3, P_5, and P_4, P_6, each equal and parallel to P_1 and P_2, as indicated in Figure; then

P_1 and P_6 give $2P_1$ at O, acting *upwards*,

P_2 and P_5 give $2P_1$,, ,, *downwards*;

$\therefore$ P_1, P_2, P_5, and P_6 equilibrate,

and may be suppressed (Art. 33), leaving P_3 and P_4 forming a couple = original couple (being of equal arm and force).

88. **Some Properties of Couples.**—It follows from Arts. 85, 86, and 87, that—

1°. A couple may be moved parallel to itself without altering its effect.

2°. Two couples of equal moment are equal when they are in the same plane, or in parallel planes.

3°. A couple may be turned in its own plane through any angle about any point in its arm without altering its effect.

4°. A given couple (Pp) may be replaced by another having any given line a in its plane for arm. The new couple will be

$$\left(\frac{Pp}{a} \cdot a\right).$$

89. **Composition of Coplanar Couples.**—Any number of coplanar couples are equivalent to a single couple, whose moment = algebraic sum of moments of couples.

For, let L, M, N, &c., represent the moments of the couples. The couples may be reduced to a set of couples with a common arm x, and with forces at its ends

$$\frac{L}{x}, \quad \frac{M}{x}, \quad \frac{N}{x}, \quad \text{&c.}$$

of same lines of action. These forces are equivalent to single forces at the ends of x

$$= \frac{L + M + N + \text{&c.}}{x}$$

of opposite direction, which constitute a single couple whose

$$\text{Moment} = L + M + N + \text{&c.};$$

$$\therefore \quad \text{&c.} \qquad \text{Q. E. D.}$$

Cor. 1.—Since axis of resultant couple is proportional to $L + M + N + \text{&c.}$, it follows that coplanar couples may be compounded by adding (algebraically) their axes.

Cor. 2.—Couples in parallel planes, being reducible to coplanar couples, may be similarly compounded.

DEF.—Intersecting couples are those whose planes intersect.

90. Composition of two Intersecting Couples.

Let the couples be reduced to equivalent couples, with a common arm r along line of intersection of their planes, and let the forces of the reduced couples be P and Q, and R their resultant; then resultant couple $= Rr$, and its plane is that of the R's at ends of r.

91. Couples Compounded by Compounding their Axes.—If OA and OB be the axes of two intersecting couples, OC is the axis of their resultant couple. (Fig. 4, p. 14.)

For the couples being reduced, as in Art. 90, OA, OB, and OC are the sides and diagonal of a parallelogram, similar to that of which P, Q, and R are sides and diagonal.

Cor. 1.—From this Article, it appears that all theorems true of the composition of statical forces are equally true of the composition of the axes of couples.

EXERCISES.

1. Two couples, whose moments are 3 and 4, respectively, act in planes at right angles to each other; determine the resultant couple.

Draw from any point in the line of intersection of the planes perpendiculars 3 and 4, one to each plane. The diagonal 5 of the parallelogram determined by these two perpendiculars is the axis of the resultant couple.

2. The plane of a couple makes angles α and β with two planes at right angles to each other. Show that if L = moment of couple, that the couple may be replaced by two couples whose moments $= L \cos \alpha$ and $L \cos \beta$, respectively.

3. Two couples, whose moments are L and M, are in planes inclined at an angle ϕ. Show that if G = moment of resultant couple that

$$G^2 = L^2 + M^2 + 2L \cdot M \cos \phi.$$

4. Three couples, whose moments $= L$, M, and N, respectively, have their planes mutually at right angles. Show that if G = moment of resultant couple; then

$$G = \sqrt{L^2 + M^2 + N^2}.$$

5. Show that any couple of moment G may be replaced by three couples of moments $= G \cos \alpha$, $G \cos \beta$, $G \cos \gamma$, respectively, whose planes are inclined to the plane of the original couple at angles α, β, and γ, respectively.

CHAPTER VIII.

CENTRE OF PARALLEL FORCES.

N.B.—By the distance of a force from a line or plane is meant the distance of its point of application from the line or plane.

92. **Magnitude, Direction, and Point of Application of the Resultant of a System of Parallel Forces, acting on a Rigid Body.**

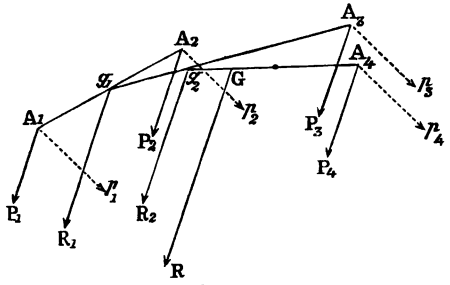


Fig. 26.

Let P_1, P_2, P_3, &c., be parallel forces, acting at A_1, A_2, A_3, &c., points of a rigid body; then R_1 (resultant of P_1 and P_2) $= P_1 + P_2$, and acts at g_1, so that

$$P_1 \times A_1 g_1 = P_2 \times A_2 g_1. \quad \text{(Art. 73)}$$

Similarly, resultant of R_1 and P_3, viz. $R_2 = R_1 + P_3 = P_1 + P_2 + P_3$, and acts at g_2, so that $R_1 \times g_1 g_2 = P_3 \times A_3 g_2$; *i. e.* so that

$$(P_1 + P_2)\, g_1 g_2 = P_3 \times A_3 g_2.$$

In the same way, by combining R_2 with P_4, and so on, we finally obtain a point G, at which the resultant R of all the forces acts;

then $R = P_1 + P_2 + P_3 + P_4$, or, shortly, $\Sigma(P)$.

The point G is termed **The Centre of Parallel Forces,** or simply **The Centre.**

It is so called, because it depends *solely* on the magnitudes and points of application of the forces, and *not* on their common direction. For example, if the forces were all turned through a common angle, so as to act in the direction indicated by the dotted lines in the diagram, the point G would remain unaltered. It is also evident that the position of G depends on the relative, and not the absolute, magnitudes of P_1, P_2, P_3, &c.

Remarks on Art. 92.

In the diagram, and the proof of Art. 92, we have assumed the forces to be *like parallel forces.* If this be not the case, the point G may still be found by the help of Art. 74, except in the particular case where the whole system of forces gives rise to a couple, in which case there is no centre (cf. Art. 79).

93. **Proof that**

$$P_1x_1 + P_2x_2 = (P_1 + P_2)\bar{x};$$

where P_1 and P_2 are two like parallel forces, x_1 and x_2 their distances from any line (L) in their plane, and $\bar{x}$ the distance of their centre from L.

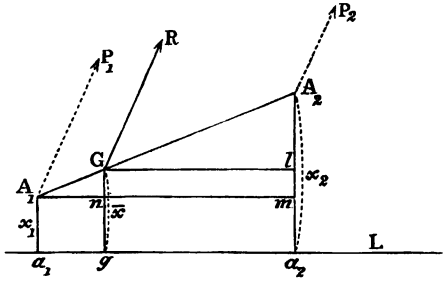


Fig. 27.

$$P_1 \times A_1G = P_2 \times A_2G; \quad \text{(Art. 73)}$$

$$\therefore \frac{P_1}{P_2} = \frac{A_2G}{A_1G};$$

Therefore $$\frac{P_1+P_2}{P_2}=\frac{A_2G+A_1G}{A_1G}=\frac{A_1A_2}{A_1G}=\frac{A_2m}{Gn}.$$

But $$A_2m=x_2-x_1.$$

And $$Gn=\bar{x}-x_1;$$

$$\therefore\quad \frac{P_1+P_2}{P_2}=\frac{x_2-x_1}{\bar{x}-x_1};$$

$$\therefore\quad (P_1+P_2)\,\bar{x}-(P_1+P_2)\,x_1=P_2x_2-P_2x_1.$$

Whence $$P_1x_1+P_2x_2=(P_1+P_2)\,\bar{x}.$$

Cor. 1.—If P_1, P_2, P_3, &c., be any number of coplanar parallel forces; x_1, x_2, x_3, &c., their distances from *any line* (L) in their plane; $\bar{x}$ the distance of their centre from L; then

$$P_1x_1+P_2x_2+P_3x_3+\&c.=(P_1+P_2+P_3+\&c.)\,\bar{x},$$

or $$\Sigma(Px)=\Sigma(P).\bar{x}.$$

Cor. 2.—In *Cor.* 1, if the forces be parallel, but not coplanar, and if x_1, x_2, x_3, &c., denote their distances from *any plane*, and $\bar{x}$ the distance of their *centre* from the same plane; then, as before,

$$\Sigma(Px)=\Sigma(P).\bar{x}.$$

Cor. 3.—A system of parallel forces can only have *one centre*.

For $$\bar{x}=\frac{\Sigma(Px)}{\Sigma(P)}.$$

Now, if in *Cor.* 1 we take *two* intersecting right lines, and, in *Cor.* 2, *three* mutually intersecting planes, the distances of G from these, given by above formula, determine in both cases only one single point; and therefore, &c.

N.B.—The above formulae are true, where x_1, x_2, &c., and $\bar{x}$, are lines equally inclined to the line or plane.

Cor. 4.—If the forces in *Cors.* 1 and 2 be in equilibrium, *i.e.* such as to give neither a resultant force nor a resultant couple; then

$$\Sigma(Px)=0;$$

i.e. the sum of the products obtained by multiplying each force by its distance from the line or plane = 0.

94. **Centre of Gravity.***

If the points A_1, A_2, A_3, &c., of Art. 92, Fig. 26, p. 47, be indefinitely close to each other, and if we conceive particles of matter situated at them forming a continuous body, then the centre of parallel forces of the weights (P_1, P_2, P_3, &c.) of these particles is termed the centre of gravity of the body. The effect of altering the position of the body in any manner is simply to turn the weights P_1, P_2, &c., round their points of application through some common angle, and this does not alter the position of G (Art. 92).

We may therefore define the C. G. of a body thus:—

Definition (1°) **of C. G. of a Body.**—The C. G. of a body is that point through which will pass *in every position of the body* the resultant of all the forces which, in consequence of gravity, act upon its particles.

Weight.—The resultant of all the elementary forces acting on a body in consequence of gravity is equal to their sum, and is termed the *weight of the body*.

If we may assume all matter to be resolvable into an indefinitely great number of homogeneous elementary molecules, to which conclusion the permanence of weight through every chemical modification of matter plainly points, we may thus define the C. G. of a body.

Definition (2°) **of C. G. of a Body.**—The C. G. of a body is the centre of parallel forces of a system of equal and *like* parallel forces acting on the ultimate molecules of which the body is composed.

The C. G. of a System of Heavy Bodies is the centre of parallel forces of a set of parallel forces proportional to the weights of the bodies, and acting through their centres of gravity.

Cor. 1.—**A Body, or System of Bodies, can have only one Centre of Gravity.**—For (*Cor.* 3, Art. 93) a system of parallel forces has only one centre, and ∴ &c.

* C. G. means Centre of Gravity.

95. **Important Centres of Gravity.**

1. Straight line, ...	Middle point.
2. Circumference or area of a ⊙,	Centre.
3. Perimeter or area of a ▭,	Intersection of diagonals.
4. Volume or surface of a sphere,	Centre.
5. Right circular cylinder, . .	Middle point of axis.
6. Parallelopiped, . . .	Intersection of diagonals.

These are evident, the figures in question consisting of equal parts equidistant from the points in question.

The lines, surfaces, and volumes, are supposed to be homogeneous and material.

Homogeneous bodies are those in which matter is uniformly distributed, and consequently are such that equal volumes are of equal weights.

96. **The C. G. of two Homogeneous Bars.**—Let P and Q be the weights of the bars. Join AB (middle points

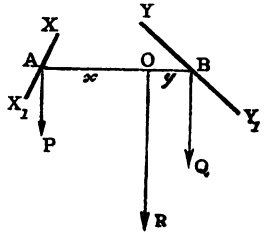


Fig. 28.

of bars). Then if O be C. G., we have

$$x + y = AB \text{ (given).} \qquad (1)$$

$$Px = Qy; \qquad (2)$$

whence x and y, and $\therefore$ O, can be found.

EXERCISES.

1. If $P = 4$ lbs., and $Q = 8$ lbs., and $AB = 24$ inches; find O. Here

$$x + y = 24,$$

$$4x = 8y;$$

whence $y = 8$ inches, which gives O.

2. If $P = 2$ lbs., and $Q = 6$ lbs., $AB = 36$ inches; find O.

Ans. 27 in. from A.

97. **Given the Centres of Gravity of two parts of a Body to find C. G. of whole.**—Let A and B (fig. 28) be the Centres of Gravity of parts. Then O, as determined in Art. 96, is the C. G. required.

EXERCISES.

1. Find C. G. of body formed of two circles P and Q, of radii 2 and 4 inches, respectively, touching each other externally. (Area of a $\odot = \pi r^2$).

Ans. C. G. is $4\frac{2}{3}$ in. from P's centre.

2. In Ex. 1, if circles were changed to spheres, find C. G. (vol. of sphere $= \frac{4}{3}\pi r^3$).

Ans. $5\frac{1}{3}$ in. from P's centre.

98. **Given C. G. of a Body, and C. G. of part thereof, to find C. G. of remainder.**

Let O (fig. 28) = C. G. of body (weight R).
,, A ,, = ,, part (weight P).
,, B ,, = ,, remaining part (weight $R - P$).

Then
$$x = AO \text{ (given)}, \qquad (1)$$

$$Px = (R - P)y, \qquad (2)$$

whence y, and therefore B is determined.

EXERCISES.

1. From a uniform circular disc (4 inches in radius) another is cut (2 inches in radius), so that their rims touch internally. Find C. G. of part left.

Ans. C. G. is $\frac{2}{3}$ in. distant from centre of larger disc.

2. In Ex. 1 change circles to spheres, and find C. G. of part left.

Ans. $\frac{4}{7}$ in. from centre of larger sphere.

3. From a square lamina whose side = 4 inches, another is cut having a common corner and diagonal, and whose side = 2 inches; determine C. G. of area left.

Ans. $\frac{\sqrt{2}}{3}$ in. from centre of larger square.

4. In 3, change the squares to cubes, and find C. G. of volume left.

Ans. $\frac{\sqrt{3}}{7}$ in. from centre of larger cube.

99. **The C. G. of a Triangle.**—By a triangle is here meant a *triangular lamina of uniform thickness and density.*

Let $\triangle ABC$ be divided into an indefinitely great number of strips, by lines parallel to base AC. The C. G. of each strip is at its middle point. All the middle points of strips lie on BQ, bisecting AC at Q;

$\therefore$ C. G. lies on median BQ.

Similarly, ,, ,, ,, AP.

Their intersection G is therefore required C. G.

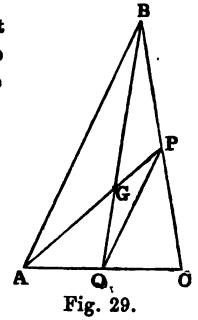


Fig. 29.

To show that $AG = \frac{2}{3} AP$.

$$\frac{AB}{PQ} = \frac{AC}{QC} = \frac{2}{1}; \qquad (1)$$

Again, $\triangle$s AGB and PGQ are similar.

$$\therefore \frac{AG}{PG} = \frac{AB}{PQ};$$

$$\therefore \quad \frac{AG}{PG} = \frac{2}{1};$$

$$\therefore \quad \frac{AG}{AG+PG} = \frac{2}{2+1} = \frac{2}{3};$$

$$\therefore \quad \frac{AG}{AP} = \frac{2}{3}.$$

$$\therefore \quad AG = \tfrac{2}{3}AP.$$

Also $$PG = \tfrac{1}{3}AP.$$

The C. G. of a triangle is thus on the median $\frac{2}{3}$ of its length from vertex.

100. The C. G. of a Polygonal Area.

Divide it into triangles, as shown, and let A, B, C, &c., be centres of gravity of triangles.

Divide AB at O, so that

$$\Delta A \times AO = \Delta B \times BO.$$

Join O to C, and find G, so that

$$(\Delta A + \Delta B) \times OG = \Delta C \times CG.$$

Then G is required C. G.

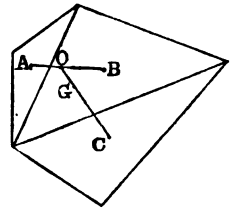


Fig. 30.

EXERCISES.

1. Show that the C. G. of three equal heavy particles placed at the vertices of a triangle coincides with the C. G. of the area of the Δ.

2. Show that the centres of gravity of a system of triangles, on the same base and between the same parallels, lie on a straight line parallel to the common base.

3. Show that the centres of gravity of any Δ and of the Δ formed by joining the middle points of its sides coincide.

4. **C. G. of a Trapezium.**—If a and b denote the parallel sides of a trapezium, show that the C. G. of the trapezium lies on the line joining the middle points of a and b, and divides this line in the ratio

$$\frac{2a+b}{2b+a}.$$ (Archimedes.)

5. A triangular lamina is submerged beneath a fluid, so that its vertices are distant 8, 9, and 10 inches, respectively, from the surface; determine the depth of its C. G. beneath the surface. *Ans.* 9 inches.

101. The C. G. of the Perimeter of a Plane Triangle.

The weights of the sides of $\Delta\ ABC$ act at their middle points, and are proportional to a, b, and c, the lengths of these sides, respectively.

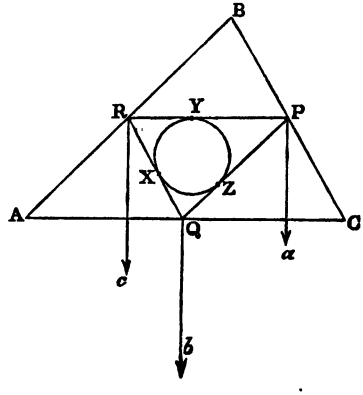


Fig. 31.

Then resultant of a and b divides PQ at Z, so that

$$\frac{PZ}{QZ} = \frac{b}{a}. \quad \text{(Art. 73.)}$$

Also,
$$\frac{PR}{QR} = \frac{b}{a};$$

because
$$\left(PR = \frac{b}{2}, \text{ and } QR = \frac{a}{2}\right);$$

$$\therefore \quad \frac{PZ}{QZ} = \frac{PR}{QR};$$

$\therefore$ RZ bisects $\angle R$. (Euc. VI. 3.)

But resultant of weights of a and b acts at Z, and weight of c acts at R;

$\therefore$ C. G. lies on bisector RZ.

Similarly, C. G. ,, ,, PX;

$\therefore$ the required C. G. is at their intersection, and this is **centre of circle inscribed in** PQR.

102. The C. G. of a Triangular Pyramid or Tetrahedron.—Resolve the solid into thin slices by planes parallel to base ABC. Each slice (abc) is a triangle similar to base. Join vertex D to C. G. of base, viz. E. This line DE passes through C. G. of each slice,* and therefore C. G. of

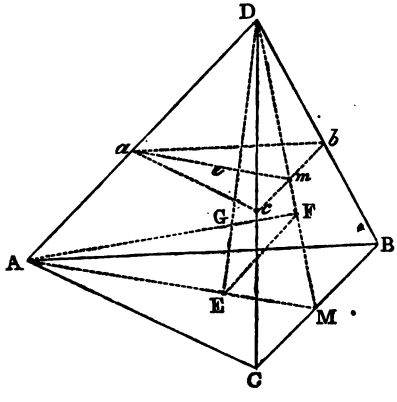


Fig. 32.

whole solid lies on DE. Similarly it lies on AF, joining A to C. G. of CDB; therefore centre of gravity is at intersection (G) of these two lines. To show that

$$EG = \tfrac{1}{4}ED$$

(*i.e.* $\frac{1}{4}$ distance from C. G. of base to opposite vertex).

The triangles ADG and EGF are similar;

$$\therefore \quad \frac{AD}{EF} = \frac{DG}{GE}.$$

But
$$\frac{AD}{EF} = \frac{AM}{EM} = \frac{3}{1} \text{ (Art. 99)};$$

$$\therefore \quad \frac{DG}{GE} = \frac{3}{1}; \quad \therefore \quad \frac{GE}{GE+DG} = \frac{1}{1+3} = \frac{1}{4}$$

$$\therefore \quad \frac{EG}{ED} = \frac{1}{4}; \quad \therefore \quad EG = \frac{1}{4}ED.$$

---

* This easily appears by remembering that the parallel planes ABC and abc are cut in pairs of parallel lines by every plane which intersects them.

Cor.—**The C. G. of a Cone or Pyramid** is on the line joining the vertex and the C. G. of the base, and is at a point distant one-fourth of the length of this line from the base (*measured on this line*).

EXERCISES.

1. A right cone is cut out of a cylinder having the same base and height: show that the C. G. of the remainder is $\frac{1}{4}$ of the height of the cone above the base.

2. Out of a right cone a similar one is cut in such a way that their axes are in the same right line, and their bases in the same plane; show that the height of the C. G. of the remainder above the common base is $\frac{1}{4}\frac{h^4 - h_1^4}{h^3 - h_1^3}$, where h and h_1 are the respective heights of the original cone and of that cut away.

103. **Property of C. G.**—*If a body suspended from a fixed point* (S) *be at rest under the influence of gravity alone, its* C. G. (*viz.* G), *must lie in the vertical line through the point of suspension.*

Two forces only act on the body. Its own weight acting vertically downwards through G and the reaction of S. These forces balance, and are therefore equal and opposite (Art. 33); therefore, &c.

Remarks.—If the body be suspended from S *by means of a string*, G must be *vertically below* S; but if S be *rigidly connected* with the body, G may be either vertically *above* or *below* S. If S coincide with G, the body will rest indifferently in any position.

104. **If the C. G. of a Body be fixed it will Balance in any Position.**

For in every position of the body its weight is balanced by the reaction of the point.

105. **Stable, Unstable, and Neutral Equilibrium.**—A body at rest is in stable equilibrium if, when slightly disturbed, it *tends to recover* its former position; in unstable equilibrium if, under the same circumstances, it *tends to recede* further from its original position; and in neutral equilibrium if it *remains at rest* in its new position.

The Equilibrium of a Body, suspended from a fixed Point, and under the influence of Gravity alone, is

stable	when its C. G. is	below	point of suspension.
unstable		above	
neutral		at	

Let C. G. be G, and point of suspension S.

1°. Let the body initially occupy the position in which G is vertically *below* S, and let it be moved into the position represented (Fig. 32). The weight GC, being resolved into GA and GB, respectively along and ⊥ to

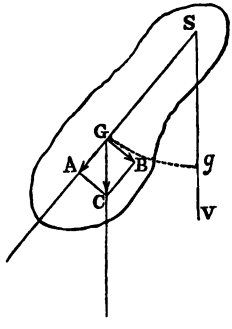


Fig. 33α.

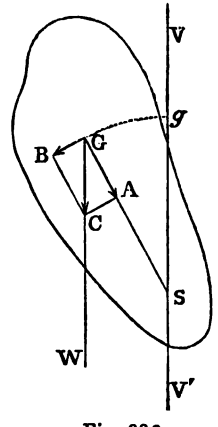


Fig. 33β.

SG, the component GA is destroyed by S, and GB remains, which tends to move the body back to its initial position. Equilibrium is therefore *stable*.

2°. Let G initially be *above* S (Fig. 33). In this case GB tends to move G *away from* its initial position. Equilibrium is therefore *unstable*.

3°. When G coincides with S the whole weight is sustained in every position by S. The body therefore rests indifferently in any position, and equilibrium is *neutral*.

N.B.—If on SV we take $Sg = SG$, it will be observed that in Fig. 33α, G is *raised* by the displacement of the body, and in Fig. 33β, *lowered*.

A right cone resting on a horizontal plane is in stable, unstable, or neutral equilibrium, according as it rests on its base, vertex, or slant side.

106. Experimental Determination of C. G.

Suspend the body from any two points in succession, and mark *relatively to the body* the two verticals through these points. These lines each pass through the C. G., and therefore their intersection is the C. G. By this means the C. G. of an irregularly-shaped body may be determined.

107. Every System of Heavy Particles, whether forming a Continuous Body, or not, has one, and only one C. G.

For, suppose G and G_1 are two centres of gravity. Turn the system about, so that $G\ G_1$ is a horizontal line. Then the resultant of the weights acts in a vertical line through both G and G_1, which is absurd; therefore, &c.

108. Base of a Body in Contact with a Plane Surface.

When a body is placed on a plane surface it touches it in a number of points. Let these points of contact be so joined by straight lines as to form a *polygon having no re-entrant angle, and including within its area all the points of contact.* This polygon is the base of the body. For example, a table having three legs situated near its edge, and one in its centre, will have for its base the triangle determined by the outer legs, provided the central leg lies *within* the area of this triangle.

109. A Body placed with its Base upon a Plane Surface will Stand or Fall, according as the Vertical Line through its C. G. falls within or without the Base.

Let $ABCD$ (Fig. 34) be the body resting on a plane surface, and let $AxyzB$ denote the points of contact with plane. There will be developed at these points reactions the resultant of which will, if possible, take such a magnitude and direction as to destroy W (*i. e.* body's weight). Hence this resultant will act at r, where vertical through g_1 intersects base, and will be equal and opposite to W. If, now, the body be loaded towards C, this will have the effect of moving the C. G. towards the right. The total reaction will follow it, and will, in all cases, if possible, take a value equal and opposite to the weight of the body. This is always possible as long as the vertical through C. G.

of body falls within base. Let the C. G. be at g_2, vertically above B, the edge of the base. In this case the body will be *just* in equilibrium.

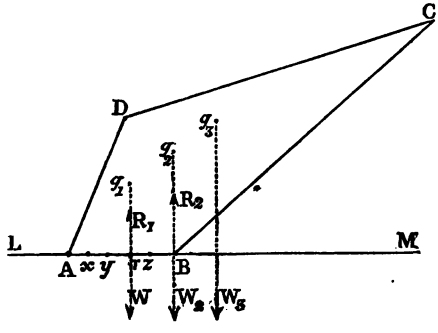


Fig. 34.

Let C. G. move to g_3, where vertical through it falls *without* the base.

In this case the body will topple over about B, as the total reaction *cannot* move further to the right than B, and therefore cannot counteract W_3.

110. **Measure of Stability of Equilibrium.** — A body's position of equilibrium is more or less stable according

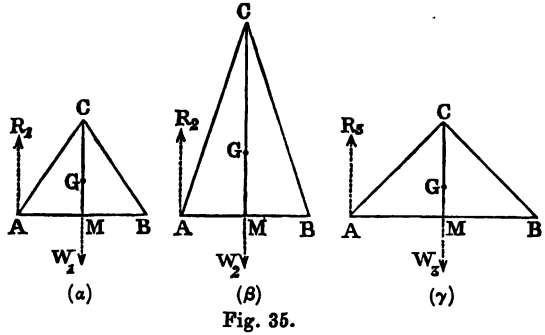


Fig. 35.

as the angle is greater or less through which the body must be moved before it ceases to tend to return to its original position. Thus, for example, a common brick is in a more or less stable position according as it is placed standing on its end or resting on its long flat side.

In the preceding diagrams (Fig. 35, α, β, γ) the $\angle$ GAR may be taken as a measure of stability, where G = C. G. of body, A = edge of base, AR = vertical line through A. For this is the $\angle$ through which the body must be turned about A before W passes through *edge of base*.

From the diagrams it appears that **other things being equal, the lower the C. G. the greater the stability** (*vide* Figs α and β, in which the bases are equal). Also the greater the base, the greater the stability (Figs. α and γ, in which the vertical heights of G are equal).

N. B.—The principles of Arts. 109 and 110 are true, whether the body rests on a horizontal or inclined plane, if only the body on the inclined plane be prevented from slipping by friction (Chap. XI.). When this is the case the body will not topple over by an increase of the inclination of the plane until the line GA be vertical, *i. e.* until the inclination of plane be equal to the angle GAR, as shown in the annexed diagram.

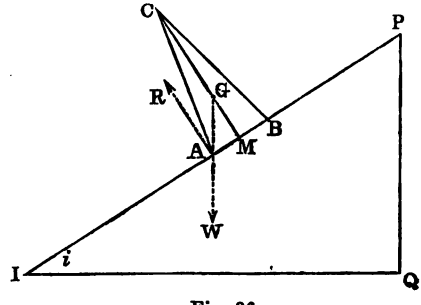


Fig. 36.

EXERCISES.

[*Exercises marked thus * may be omitted till* CHAPTER XI. *is read.*]

* 1. A body in shape a sphere is loaded in such a manner that its C. G. is not at its geometrical centre. Determine its positions of stable and unstable equilibrium when it is placed on a smooth horizontal plane.

Ans. Stable C. G. vertically below geometrical centre.
Unstable ,, above ,, ,,

2. A short circular cylinder of wood has a hemispherical end. When placed with its curved end on a smooth table it rests in any position in which it is placed. Determine the position of its centre of gravity.—(*London Matriculation.*)

Ans. At the centre of the sphere of which the hemispherical end is one-half.

*3. An equilateral triangle stands on a perfectly rough inclined plane (*vide* Art. 143); determine the ratio of the height to the base of the plane when the triangle is on the point of overturning. *Ans.* $\sqrt{3}:1$.

*4. A heavy cube rests on a perfectly rough plane of small inclination with one edge parallel to the intersection of the plane with the horizon. If the inclination of the plane be gradually increased, determine its magnitude when the cube overturns. *Ans.* 45°.

*5. If, in Ex. 4, the plane be not perfectly rough, determine the coefficient of friction when the cube begins to slip, and overturn at the same instant. *Ans.* 1.

*6. A cone whose vertical angle is 2α is placed on a perfectly rough plane of small inclination. If the inclination be gradually increased, determine its magnitude when the cone overturns.

Ans. If i be the inclination, $4 \tan i = \tan \alpha$: cf. Chap XI., Ex. 10.

General Exercises on Centres of Parallel Forces, and of Gravity.

1. A uniform rod, 6 feet long, and weighing 12 lbs., has weights of 4 lbs. and 8 lbs. suspended from its ends. Find the C. G. of the loaded rod. *Ans.* .6 inches from middle of rod.

2. Four heavy particles, whose weights are 1, 3, 5, and 7 lbs., are placed along a weightless rod, 6 feet in length, so as to divide it into three equal parts; required the distance of the C. G. from the middle point of the rod. *Ans.* $1\frac{1}{4}$ feet.

3. A Δ suspended from one of its vertices has its base horizontal; show that it is isosceles.

4. The sides of a plane Δ are 3, 4, and 5, respectively; show that if it be suspended from the centre of its inscribed $\odot$ it will rest with the side 3 horizontal.

5. A heavy Δ ABC is suspended successively from the angles A and B, and the two positions of any side are found to be at right angles; prove that $5c^2 = a^2 + b^2$.

6. If in the Δ of Ex. 4, the inscribed $\odot$ be removed; find the C. G. of the remainder.

Ans. Its distances from the sides 3 and 4 are, respectively, $\frac{8-\pi}{6-\pi}$, and 1.

7. The sides of a Δ ABC, right-angled at C, are, respectively, a, b, and c. If $a = 2b$; find the $\angle$ between the two positions of b, when the Δ is successively suspended from A and C. *Ans.* $45° + A$.

8. A triangular slab of uniform thickness and density is supported on props at its three vertices. Prove that the pressures on the props are all equal.

9. A square table stands on four legs, placed respectively at middle points of its sides; find the greatest weight that can be placed at one of the corners without upsetting the table.—(*Woolwich.*) *Ans.* W (the weight of table.)

10. Squares are described on the three sides of a right-angled isosceles triangle, on the opposite sides of each to that at which the triangle lies. Find the C. G. of the whole figure so formed.—(*The Previous Cambridge.*)

Ans. If a = length of the two equal sides; then C. G. is distant

$\frac{a}{2}$ from each of the two equal sides.

11. Squares are described on two of the sides of an equilateral triangle on the opposite side of each to that on which the triangle lies. Find the C. G. of the whole figure so formed.—(*The Previous Cambridge.*)

Ans. If a $\perp$ be drawn from the vertex to the base of the Δ, the C. G. lies on this $\perp$ at a distance from base $= a\frac{4\sqrt{3}+5}{16+2\sqrt{3}}$.

12. Prove that a system of heavy particles has one, and only one, C. G. A straight rod, one foot in length, and weighing one ounce, has an ounce of lead fastened to it at one end, and another ounce of lead fastened to it at a distance from that end equal to one-third of its length; find its C. G.—(*Woolwich.*) *Ans.* $3\frac{1}{3}$ in. from loaded end.

13. Find the C. G. of three weights placed at the centres of the $\odot$s escribed to a triangle, and inversely proportional to the radii of those circles.—(*Woolwich.*) *Ans.* Centre of inscribed circle.

14. $ABCD$ is a square lamina; if the $\odot$ inscribed in the ΔABC is cut out; find the C. G. of the remaining area.—(*Woolwich.*)

Ans. Distance of C. G. from centre of square is $\frac{\pi(5\sqrt{2}-7)a}{\sqrt{2}\{2-\pi(3-2\sqrt{2})\}}$, where a is side of square.

15. A, B, C, D, are the angles of a parallelogram taken in order; *like* parallel forces 6, 10, 14, 10 act at A, B, C, D, respectively; show that the centre and resultant of the parallel forces will remain unchanged if, instead of these forces, the parallel forces 8, 12, 16, 4 act at the points of bisection of the sides AB, BC, CD, DA, respectively.—(*Woolwich.*)

16. Equal and *like* parallel forces act at seven of the corners of a cube, whose diagonal is a, the remaining corner having no force acting at it. Determine the centre of parallel forces.

Ans. $\frac{4a}{7}$ distant from the unoccupied angle.

17. Find the C. G. of particles placed at the angular points of a Δ—(1) When the particles are equal; (2) when they are proportional to the opposite sides of the Δ.—(*Woolwich.*)

Ans. (1) Intersection of medians; (2) centre of inscribed $\odot$.

18. Show that the C. G. of a triangular pyramid is the same as that of four equal particles placed at its vertices.

19. Find the C. G. of a solid triangular pyramid whose faces are equilateral triangles. Show that the position of the C. G. for the four faces considered as plane areas will be the same as it is for the solid pyramid.—(*Woolwich.*)

20. If $ABCD$ be a tetrahedron, and if the plane CDE, passing through the edge CD, cuts AB in E; prove that the line joining the centres of gravity of the tetrahedrons $ABCD$ and $AECD$ is parallel to AB.—(*Woolwich.*)

21. A solid in the form of a right circular cone has its base scooped out, so that the hollow formed is a right cone on the same base, and of half the height of the original cone; find the position of the C. G. of the solid so formed.—(*Woolwich.*) *Ans.* $\frac{3}{8}$ height from base.

22. A body consists of two parts, A and B, whose centres of gravity are at the points P and Q, respectively. The part B is moved, so that its C. G. comes to Q', the part A remaining fixed; prove that the C. G. of the whole body moves through a distance parallel and proportional to QQ'.—(*Woolwich.*)

23. If, in the last Exercise, G and G' be the two positions of the C. G. of the whole body, show that $(A + B)\ GG' = B.QQ'$.

111. **Important Principle.**—Exercises 22 and 23 give us the following important principle:—If from any body M a part B be detached, and moved *along any path* into a new position; then $B \times$ length of path which its C. G. describes $= M \times$ length of the path its C. G. describes owing to the motion of B.

EXERCISES.

1. If a portion m of any mass M is moved to any new position, show that the C. G. of the entire mass is thereby moved in a direction parallel to the displacement of the C. G. of m, and over a distance $= \frac{m}{M}.D$, where D = above distance between the two positions of the C. G. of m.—(*Woolwich*).

2. A triangular piece of paper is folded across the line bisecting two sides, the vertex being thus brought to lie on the base. Find the C. G. of the paper in this position.—(*Woolwich.*)

Ans. Drop a perpendicular from G, the C. G. of original triangle, on its base. The C. G. of paper in new position lies on this line, and is distant from G by $\frac{1}{12}$th altitude of original triangle.

CHAPTER X.

CONDITIONS OF EQUILIBRIUM.

112. Caution about the Resolution of Forces.

As long as there is only *one* force to resolve into components, there is no ambiguity about the resolution to be effected.

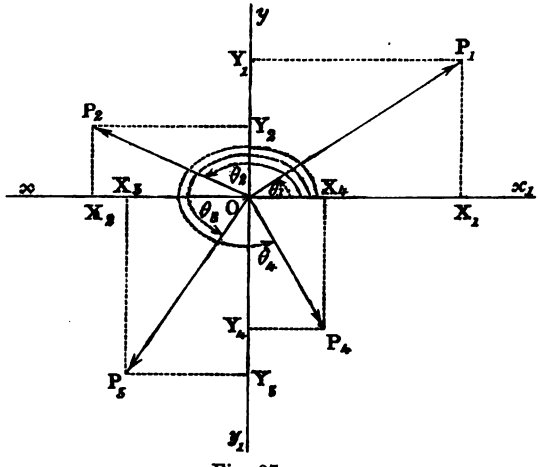


Fig. 37.

In Art. 51, $X = F\cos\theta$, where θ is the *acute* angle F makes with Ox. But when we have a number of forces to resolve, we must be careful to measure our angles from the same part of the axis. In the adjoining diagram (Fig. 37) the angles are measured from ox_1 (as shown), and the resolved parts of P_1, P_2, P_3, and P_4, along xx_1 are, respectively,

$$P_1\cos\theta_1,\ P_2\cos\theta_2,\ P_3\cos\theta_3,\ \text{and}\ P_4\cos\theta_4.$$

By using these angles the components take of themselves the proper signs Thus $P_2\cos\theta_2$ and $P_3\cos\theta_3$ are both $-$, $P_1\cos\theta_1$ and $P_2\cos\theta_2$ both $+$.

In practice, however, we can generally *see* how the components will act, and we can then use the acute angles P_2Ox, and P_3Ox, instead of θ_2 and θ_3, and attach the proper signs (*vide* the Inclined Plane, Art. 171, *sqq.*).

113. Conditions of Equilibrium of a System of Coplanar Concurrent Forces.*

1°. *The sum of the resolved forces in any direction* = 0.

2°. *The sum of the resolved forces at right angles to the former set* = 0.

For if X and Y denote the algebraic sums of the forces resolved along xx_1 and yy_1, respectively (Fig. 37), then R being the resultant of these forces,

$$R = \sqrt{X^2 + Y^2} \quad \text{(Art. 50.)}$$

If now the forces be in equilibrium,

$$R = 0.$$

This implies both

$$X = 0, \text{ or } \Sigma(P\cos\theta) = 0, \text{ giving condition 1°,}$$

and

$$Y = 0, \text{ or } \Sigma(P\sin\theta) = 0, \quad ,, \quad ,, \quad 2°.$$

EXERCISES.

1. Show that if in Fig. 37, P_1, P_2, P_3, and P_4 be = 10, 5, $5\sqrt{3}$, and 10, respectively, and be inclined to Ox_1 at angles of 60°, 120°, 210°, and 300°, respectively, they will equilibrate.

2. Show that forces = 2, 1, and $\sqrt{3}$, inclined to Ox_1 (Fig. 37) at angles of 30°, 150°, and 240°, respectively, equilibrate.

114. A Force replaced by a Force and Couple.

A force acting at any point (A) (Fig. 38, p. 67) may be replaced by an equal and like parallel force acting at any other point (O), and a couple.

For, let P act at A: introduce at O + P and − P, which does not affect P's action.

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* Parallel forces are concurrent forces meeting at infinity.

Then P at A, and $-P$ at O, give a couple Pp, leaving $+P$ at O; $\therefore$ P at A is equivalent to P at O and the couple Pp.

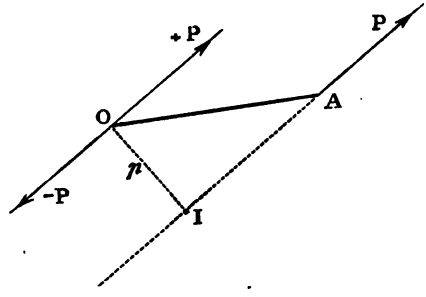

Fig. 38.

115. Resultant of a Force (F) and Couple (Pp) in the same Plane is a single Force.

For $(Pp) = (F.\frac{P}{F}p)$, *i. e.* a couple whose force $= F$, and arm $\frac{P}{F}p$. This couple may be moved so that one of its forces (F, F) equilibrates the original given force (F), leaving a single force F parallel to original force, and acting at a point distant from the original point of application of the given force F by a length $= \frac{P}{F}.p$.

116. Conditions of Equilibrium of a System of Coplanar but non-Concurrent Forces.

The three conditions of equilibrium are—

1. *The sum of the resolved forces in any direction* $= 0$.
2. *Their sum in a direction at right angles with the first direction* $= 0$.
3. *The sum of the moments about any point* $= 0$.

Let P_1, P_2, P_3, &c., be the forces acting at the points A_1, A_2, A_3, &c. Consider any one force P_1.

By Art. (114), this may be replaced by P_1 at O, equal, parallel, and co-directional to P_1 at A_1, and a couple P_1p_1.

Each force being similarly resolved, we have finally a set of coplanar concurrent forces acting at O, giving a resultant force (R), and a set of coplanar

couples $P_1 p_1$, $P_2 p_2$, &c., giving a resultant couple (G). For equilibrium we must have both

$$R = 0 \text{ and } G = 0.$$

$R = 0$ gives both $X = 0$ and $Y = 0$, where X and Y have the same meanings as in Art. 113, and therefore gives conditions 1° and 2°, and $G = 0$ gives condition 3°.

117. Remarks on the solution of Mechanical Problems.

The solution of Problems in Mechanics is simplified—

1° *By resolving perpendicular to unknown reactions, or to forces with which we are not concerned.*

2° *By taking moments about points on unknown forces which thus do not appear in the equation of moments.*

EXERCISES.

1. A set of forces act on a rigid body in such a way that the lines representing their points of application, magnitudes, and directions, form a closed plane polygon. Show that this system of forces is equivalent to a couple.

2. A uniform beam rests on two smooth inclined planes whose angles of inclination to the horizon are α and β, respectively. Find the angle of inclination (θ) of the beam to the horizon and the pressures on the planes.

Ans. $\tan\theta = \dfrac{\sin(\beta - \alpha)}{2 \sin\alpha \sin\beta}$,

$$R_1 = \frac{W \sin\beta}{\sin(\alpha+\beta)},$$

$$R_2 = \frac{W \sin\alpha}{\sin(\alpha+\beta)}.$$

3. A uniform beam AB rests with one extremity A against the inside surface of a smooth hemispherical bowl whose rim remains horizontal, and supports the beam at some point (P) of its length; determine the position of equilibrium.

Ans. If $2a$ = length of beam,
r = radius of hemisphere,
$\theta = \angle$ of inclination of beam to horizon,

$$4r\cos^2\theta - a\cos\theta - 2r = 0.$$

4. A uniform beam AB rests with one end A on the inside of a smooth hemispherical bowl whose rim remains horizontal, and the other end B against a smooth vertical wall; find the position of equilibrium.

Ans. $2a$, r, and θ having the same meanings as in Ex. 3,

c = distance of centre (C) of hemisphere from vertical wall, and $\phi = \angle$ radius AC makes with horizon; then

$$2 \tan\theta = \tan\phi, \qquad (1)$$

$$2a\cos\theta = r\cos\phi + c, \qquad (2)$$

whence θ can be found.

5. A trap-door opens downwards by turning on a horizontal hinge, and is kept inclined at an angle α to the horizon by means of a cord attached to the middle point of its side opposite the hinge, and to a point in the horizontal plane through the hinge, at a distance from the hinge equal to the length of the door. Determine the tension on the cord.

Ans. Tension $= W.\dfrac{\cos\alpha}{2\cos\dfrac{\alpha}{2}}$, where W = weight of door.

6. A uniform heavy beam AB rests against a smooth peg (P), and against a smooth vertical wall AD; find the position of equilibrium of the beam, and the pressures on the wall and peg.

Ans. If $2a$ = beam's length,

W = beam's weight,

c = $\perp$ distance of peg from wall,

θ = inclination of beam to vertical,

$$\text{then } \sin\theta = \left(\frac{c}{a}\right)^{\frac{1}{3}},$$

$$\text{pressure on peg} = W\left(\frac{a}{c}\right)^{\frac{1}{3}},$$

$$\text{pressure on wall} = W.\frac{\sqrt{a^2 - c^2}}{c^2}.$$

7. A uniform beam whose length is $2a$, and weight W, is placed with one end A on a smooth horizontal plane, and the other B on a smooth plane inclined thereto at 60°. It is sustained in this position by means of a string attached to the junction of the planes (C) and the end of the beam A. If $CA = CB$, find the tension on the string.

Ans. Taking moments about intersection of reactions of planes, tension $= \dfrac{\sqrt{3}}{4}.W.$

118. **Any System of Forces acting on a Rigid Body is equivalent to a Resultant Force** (R), **and a Resultant Couple** (G).

For each force acting at any point is equivalent to an equal, parallel, and co-directional force acting at any assigned point (O) and a couple. (Art. 114.) The forces at O combine into a single force (R), and the couples into a single couple (G) (Art. 91); therefore, &c.

119. **Conditions of Equilibrium of a Body which has a Fixed Point** O.

The only necessary condition is that the sum of the moments about the point = 0.

For the forces are equivalent to a set of forces acting at O which are neutralized, and a set of couples equivalent to a single couple (G), which vanishes when the above condition is satisfied; therefore, &c.

EXERCISES.

1. $ABCD$ is a square. Forces of P, Q, and R lbs. act along AB, BC, and CD, respectively. Find the magnitude and direction of a force whose line of action is AD which will keep the square at rest.

Ans. $(P + Q + R)$ lbs. from A to D.

2. At what point must a rope of given length (a) be attached to a tree in order that a man pulling at the other end may have the greatest effect in upsetting the tree?

Ans. At a distance from ground $= \frac{a}{\sqrt{2}}$.

N.B.—Find when the moment about the foot of the tree = a maximum.

120. **Conditions of Equilibrium of any set of Forces which act on a Rigid Body in any way.**

Take any three straight lines at right angles to each other, and call them the axes of x, y, and z, respectively; then—

1. The sums of the resolved forces in the directions of the axes of x, y, and z are *separately* = 0.

2. The sums of the moments of the forces about the axes of x, y, and z are *separately* = 0.

These two virtually amount to six conditions. The proof is scarcely suitable to an elementary work, but may be supplied by the student.

121. **Definition of a Wrench.**—A force acting along a definite line and a couple in a plane perpendicular to this line form what is called a wrench. The definite line is called the *axis* of the wrench. The point of application of the force is *any* point on this axis, and is therefore not fixed.

122. **Any System of Forces acting on a Rigid Body are equivalent to a Wrench.**

For the forces may be replaced by a single force (R) acting at any assigned point, and a single couple (G), which latter may be replaced by two couples L and M, whose planes are, respectively, perpendicular and parallel to R. Of these couples the latter (M) may be compounded with R by Art. 114, so as to give a force R acting at a new point in a direction parallel to the direction of the original R, and therefore perpendicular to the plane of L. The system is therefore reduced to a single force (R) and a single couple whose axis is parallel to R, *i.e.* to a wrench.

CHAPTER XI.

FRICTION.

123. The bodies hitherto considered have been supposed smooth, *i. e.* such that the reaction between two such bodies was in all cases perpendicular to their surface, or line of contact.

No such bodies exist in nature, and we have now to consider the roughness of bodies in virtue of which they resist the sliding or rolling of other bodies upon them.

124. **Friction of two kinds.**

Friction is the resistance we experience when we attempt to move one body along the surface of another body against which it is pressed. It is of two kinds—1° slipping friction; 2° rolling friction.

1°. **Slipping Friction** is the resistance which bodies offer to the sliding of other bodies upon them.

2°. **Rolling Friction** is the resistance bodies offer to the rolling of other bodies upon them. This kind of friction would, perhaps, be better named resistance to rolling.

125. **Rolling less than Slipping Friction.**

Rolling friction is much less with the same surfaces and pressures than slipping friction.

Hence brakes are employed on railway trains to stop the carriages rapidly. Hence also to diminish friction the axles of bicycles and tricycles are sometimes made to rotate on ball bearings.

126. Two ways in which a Body may resist Sliding.

1°. By the possession of small inequalities on its surface, which act as fixed obstacles to sliding.

Through the existence of these obstacles the surfaces of the two bodies get interlocked, and an effort to make one slide on the other causes a strain in each of the surfaces, the force which resists the sliding being called friction.

2°. By adhesion.—Adhesion is distinguished from friction by being independent of the force by which the bodies are pressed together. Both friction and adhesion are analogous to shearing stress, *i. e.* the force (called *cohesion*) which resists an attempt to divide a solid by causing one part of it to slide on another.

127. Friction a Passive Resistance.

Passive resistances are forces which only come into existence through the action of other forces, and which always tend to destroy* the action of those other forces. For example, the resistance of a smooth curve or surface is a passive resistance, and will take any magnitude sufficient to destroy the forces which occasion it consistent with the strength of the material of which the curve or surface is composed. In the abstractions of rational mechanics we assume such curves and surfaces to be quite impenetrable, and consequently the reactions they engender may have any magnitude, no matter how great.

Again, the resistance of a fluid to the passage of a body through it is a passive resistance.

128. Friction is a Passive Resistance

which destroys, or tends to destroy, the slipping or rolling of one solid body on another.

129. Axiom about Passive Resistance.

Passive resistances always develop themselves *both in magnitude and direction*, so as to destroy motion, or this failing, to retard motion as much as possible.

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* Destroy here and elsewhere means neutralize. The conservation of energy shows that force cannot be annihilated.

130. The Total Resistance

of a rough curve, or surface, at any point to a body placed at that point, is the resultant of the normal reaction and the friction developed at the point under any specified circumstances.

In Fig. 39, R is resultant of N (normal reaction) and F (friction developed).

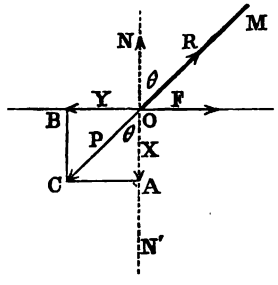

Fig. 39.

To arrive in an easy way at some important definitions and laws of friction, we shall consider the equilibrium of a weightless rod pressed in the direction of its length against a rough surface. It must be understood that in every position of the rod its end in contact with the rough surface is not a point, but a flat surface.

131. Equilibrium of a Beam pressed against a Rough Surface.

Let a beam MO (supposed weightless) be urged against a rough surface in the direction of the beam's length (Fig. 39) by a force (P) = OC. Resolve OC into components, OA perpendicular to plane, and OB parallel to plane. OA is destroyed in all cases by N (reaction perpendicular to plane), and OB will be destroyed under certain circumstances by friction (F).

Now when this *is* the case,

$$OA = P\cos\theta = N,$$

$$OB = P\sin\theta = F;$$

$$\therefore \quad \frac{F}{N} = \tan\theta;$$

$$\therefore \quad F = N\tan\theta;$$

therefore in all cases, when there is equilibrium, friction = normal reaction × tan of angle beam makes with normal.

132. **Total Resistance.**

If the resultant of N and F, Fig. 39, be taken, and called R, then R is the total resistance of the rough surface, and is equal and opposite to P, *i. e.* it acts along the beam in the direction OM.

133. **The Angle of Friction.**

Let the beam occupy the position NO, and let it be gradually moved from this position by pivoting round its end O in the plane NOM, while it is urged against the plane by a force P along its length. It will be found that the beam will not slip until the angle θ attains a certain magnitude λ dependent on the nature and state of the substances of which the beam and plane are composed. This limiting angle λ is termed *the angle of friction*, and sometimes *the angle of repose*.

134. **Definition of Angle of Friction.**

The angle of friction is the angle between the normal to a rough surface at any point, and the total resistance developed between the surface and a rough body in contact with it at that point, when the body is about to slip at that point.

135. **The Coefficient of Friction.**

We have seen that in all cases when there is equilibrium (Art. 131) $F = N.\tan\theta$.

In the position of the beam bordering on motion this becomes

$$F = N.\tan\lambda,$$

or,
$$F = \mu.N, \text{ where } \mu = \tan\lambda.$$

μ is termed the coefficient of statical friction, and may be thus defined.

136. **Definition of Coefficient of Statical Friction.**

The coefficient of statical friction is that number by which the normal reaction between two bodies must be multiplied, to give the amount of friction developed when the bodies are on the point of sliding the one on the other.

137. **The Cone of Friction**

is a right cone described round OV, the common normal to two surfaces at their point of contact, and whose semi-vertical angle = λ (the angle of friction), as shown in the diagram.

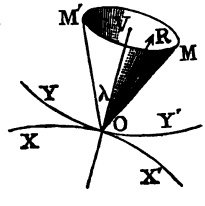

Fig. 40.

138. **The general condition that two Bodies should not slip at their point of contact** is simply that the total resistance (R) at point of contact (O) should fall within the cone of friction.

This appears from the circumstance that R can of itself assume any magnitude sufficient to prevent motion, provided its inclination to the normal is less than λ (angle of friction).

139. **Statical and Dynamical Friction** are the forces of slipping friction respectively developed by pressure between two bodies in contact with each other, prior and subsequent to the establishment of a sliding motion between the bodies.

Consider the beam in Art. 131. Statical friction developed $= N \tan \theta$, where θ = angle of inclination of P to normal, and N = normal reaction. This attains its maximum value when $\theta = \lambda$ (angle of friction). If θ is greater than λ sliding takes place, and this motion is opposed by dynamical friction, which is in general less than the maximum amount of statical friction.

140. **Caution about Friction developed.**

Statical friction may have *any* value from 0 to μN. It is only in the state bordering on motion that

$$\text{friction} = \mu N.$$

Dynamical friction is always $= \mu N$, but the coefficient μ is less for dynamical than for maximum statical friction under the same circumstances.

141. **Laws of Friction (established by experiment).**

1. Friction is directly opposed to motion, or a tendency to motion.

2. Friction is independent of the extent of the surfaces in contact.

3. Friction is directly proportional to normal pressure.

4. Friction is independent of the velocity of motion.

The friction here referred to is either maximum, statical, or dynamical friction.

Law 1 is true of all friction, no matter how developed, and is a consequence of the axiom of passive resistances (Art. 129).

142. Circumstances on which the Coefficient of Friction (μ) depends.

The coefficient of friction (μ) depends on the *pair* of bodies in contact, and not solely on one body considered by itself. Moreover, for the *same two* bodies it depends on—1°. their polish; 2°. the species and quantity of lubricant applied; 3°. the length of time the bodies have been in contact (μ increases with this time up to a certain limit).

It does *not* depend sensibly on the extent of the surfaces in contact.

Again, it is not the same for two blocks of wood, with their fibres lengthwise, as it is with them crosswise.

In fact, when great accuracy is required, μ should be determined by special experiment for the case considered.

143. Actual Values of μ.

The coefficient of friction (μ) varies greatly with the circumstances, being in some cases as low as 0·03, in others as high as 0·8. Theoretically the coefficient of friction (μ) is sometimes supposed to be capable of having any value from 0 to ∞.

A perfectly rough body is one between which and another rough body $\mu = \infty$. A Table of coefficients and angles of friction may be seen at the end of Part II.

Exercises on Friction.

1. A weight of 10 lbs. is placed on a rough horizontal plane, and it is found that it requires a horizontal force of 5 lbs. to *just* move the weight. Determine the coefficient of friction.

Ans. Friction = μ (normal reaction), *in state bordering on motion;*

$$\therefore \quad 5 \text{ lbs.} = \mu \times 10 \text{ lbs. (normal reaction)};$$

$$\therefore \quad \mu = \tfrac{1}{2}.$$

2. In Ex. 1, if force $= \dfrac{10}{\sqrt{3}}$ lbs., determine angle of friction..

Ans. $\dfrac{10}{\sqrt{3}} = \mu \cdot 10;$

$$\therefore \quad \mu = \frac{1}{\sqrt{3}} \; (= \tan 30°);$$

$$\therefore \quad \tan \lambda = \tan 30°;$$

$$\therefore \quad \lambda = 30°.$$

3. In Ex. 1, if $\mu = \frac{1}{4}$, determine horizontal force (F) necessary to move the weight.

Ans. $F = \frac{1}{4} \cdot 10; \quad \therefore \; F = 2$ lbs.

4. A weight is urged against a rough surface by forces giving a normal reaction (N) = 4 lbs., and a force of friction (F) = 3 lbs. Determine the magnitude and direction of *the total resistance* (R).

$$R^2 = N^2 + F^2 = 4^2 + 3^2 = 25;$$

$$\therefore \quad R = 5.$$

If R makes an angle θ with N,

$$\tan\theta = \tfrac{3}{4}.$$

5. If in Ex. 4 the angle of friction = 30°; find the magnitude of F for the same value of N. *Ans.* 2·309 lbs.

6. Show that the best angle of traction along a rough horizontal plane is the angle of friction.

7. A slab of any substance is placed on a rough inclined plane whose angle of inclination is gradually increased. Show that when the body begins to slip, the inclination of the plane to the horizon = the angle of friction (λ).

8. A rectangular block with a square base (side $2a$) is placed on a rough inclined plane, whose angle of inclination is gradually increased. It is found that the block begins to slide and turn round its horizontal edge at the same instant. Determine the coefficient of friction between the plane and block whose height is $2b$.

Ans. $\mu = \tan\lambda = \frac{a}{b}$.

9. Show that the initial motion of the block when it does not slide and roll at the same instant will be one of sliding or tumbling, according as

$$\mu \text{ is greater or less than } \frac{a}{b}.$$

10. A heavy right cone is placed on a rough inclined plane, the inclination of which is gradually increased. Show that the initial motion of the cone will be one of sliding or tumbling, according as

$$\mu \text{ is greater or less than } 4\tan\alpha,$$

where α is the semi-vertical angle of the cone.

11. A beam AB has its ends A and B in contact, respectively, with a rough horizontal and rough vertical plane, the beam being in a plane at once perpendicular to both planes. Determine condition of equilibrium.

Draw Aps and Aqr lines making angles = angle of friction (λ) at each side of AN (normal at A to plane).

Draw also Bqp and Brs making angles = angle of friction (λ') at B. These four lines determine a quadrilateral $pqrs$.

The condition of equilibrium then is, that vertical line through C. G. of beam should pass into the area of this quadrilateral. For the beam, if in equilibrium, is so under the action of three forces (its weight and total resistances R and R' at A and B, respectively). For equilibrium these forces must meet in a point, and if there be a point on the vertical through G (C. G. of beam), through which R and R' *can* pass, they *will* do so, and assume the requisite magnitudes to secure equilibrium; therefore, &c.

12. In Ex. 10 if the beam be uniform, find the least coefficient of friction which will allow the beam to rest in all positions. *Ans.* $\mu = 1$.

13. In Ex. 10 if the beam be not uniform, and if GA and $GB = a$ and b, respectively, where G is C. G. of beam, and if μ = coefficient of friction between beam and horizontal plane; find the least coefficient of friction between the beam and vertical plane which will allow the beam to rest in all positions.

Ans. $\mu' = \frac{a}{\mu b}$.

CHAPTER XII.

MACHINES.

144. A machine is an instrument by the agency of which a force applied at some point can be made available at some other point. Machines are considered in this Chapter in a state of rest, the question of their motion being reserved for a future chapter. (Part II., Chap. v.)

145. **Efforts and Resistances.**

Every machine is designed for the purpose of overcoming certain forces called *resistances*. The forces applied to the machine for this purpose are termed *efforts* (or powers).

146. **Power.**

The word power is ambiguous in its import. It is now almost universally employed by physicists to mean *time rate of doing work*. Electrical engineers at once think of power as so many Watts. Practical engineers think of it as so many H. P. (horse power). The context will in all cases show its meaning. (*Vide* Part. II., Chap. v.)

147. **Mechanical Advantage.**

By the mechanical advantage of any machine is meant *the ratio of the resistance to the effort* (*or power*) *when in equilibrium*.

Thus, if an effort of 10 lbs. sustain a weight of 120 lbs. by means of a machine, the mechanical advantage is 12 (*i.e.* $\frac{120}{10}$).

148. **Mechanical Powers.**—The simplest machines are called mechanical powers. They are usually considered to be six in number, and may be classified as follows:—

I. The lever, including
- the lever proper, (1)
- the wheel and axle. (2)

II. The inclined plane, including
- the inclined plane, (3)
- the wedge, (4)
- the screw. (5)

III. The pulley, ... (6)

I.—THE LEVER.

149. **Definition of a Lever.**—The lever is a rigid rod movable only about a fixed axis *F*, called the *fulcrum*.

Unless the contrary be stated, we shall assume the lever to be without weight.

The portions of the lever into which the fulcrum divides it are called the *arms* of the lever.

150. **Forces acting on a Lever.**—In the simplest cases in which the lever is employed three forces are brought into play—

(1) The effort applied to sustain or overcome the opposing resistance.
(2) The resistance or force opposing the effort.
(3) The reaction of the fulcrum.

151. **Three kinds of Lever.**

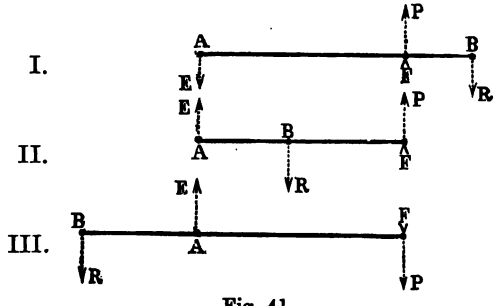

Fig. 41.

Levers are said to be of the 1st, 2nd, or 3rd kind, according as the fulcrum, resistance, or effort occupies the middle position of the three.

The diagram will enable the student to remember the three kinds of lever, and the mode of action of the forces *which act upon the lever*.

If we conceive *A* and *F* as fixed, and *B* to occupy in succession the positions in I., II., and III., we have the three kinds of lever.

152. **Instances of Levers.**

1°. **First kind.**—A crowbar used to move heavy bodies with fulcrum as in I. (Fig. 41); a poker used to raise coals in a grate; the handle of a pump. Scissors, shears, nippers, &c., consist of two levers of the first kind.

2°. **Second kind.**—A wheel-barrow, the ground being the fulcrum, the weight in the wheelbarrow the resistance, and the effort the force applied by

the workman at the handles. N.B.—The fulcrum in this case, though movable relatively to the world, is stationary as regards the workman. An oar is commonly regarded as a lever of the second kind; the fulcrum being in the water at the blade of the oar, the resistance at the rowlock, and the effort (or power) applied by the hand of the rower at the handle of the oar. But on this question see note at end of Part. I., entitled "*To what order of lever does the oar belong?*" A pair of nutcrackers is a double lever of the second kind.

3°. **Third kind.**—The limbs of animals are generally levers of this kind. For example, the human forearm used to raise a weight taken in the hand. The fulcrum is at the elbow. The effort is exerted by a muscle which, coming from the upper part of the arm, is inserted in the forearm, near the elbow, and the weight of the limb itself, together with that of the object held in the hand, constitute the resistance. A pair of tongs used to hold a coal is a double lever of the third kind.

153. Gain in Power is Loss in Despatch.

In the first and second kinds of lever the space through which the resistance is moved is *generally* less than that through which the effort moves, as the lever turns about the fulcrum, and therefore what is gained in power is lost in despatch. In the third kind of lever, however, the reverse is the case: the distance through which the resistance is moved is greater than the distance through which the effort moves, so that in this kind of lever despatch is gained at the expense of power.

154. Condition of Equilibrium of the Lever.

The necessary and sufficient condition is that the sum of the moments of the acting forces about the fulcrum = 0 (*cf.* Art. 119).

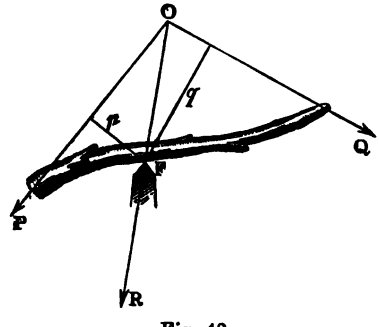

Fig. 42.

155. Pressure upon the Fulcrum of a Lever.

The pressure upon the fulcrum is in every case the resultant of the other forces acting upon the lever. *e. g.* in the adjoining diagram (Fig. 42) R (resultant of P and Q) is the pressure which the fulcrum experiences. The relation between P and Q is given by the equation $Pp = Qq$ obtained by taking moments about fulcrum (*cf.* Art. 69).

EXERCISES.

1. *Like* parallel forces of 8 and 12 lbs. act at points A and B perpendicular to a straight lever 6 feet long. Find the position of the fulcrum, and the pressure it sustains when the lever is in equilibrium.

Taking moments about the fulcrum, we obtain

$$8AF = 12BF. \quad \text{(Fig. 41.)}$$

Also $AF + BF = 6$;

whence $AF = 3\frac{3}{5}$ feet, $BF = 2\frac{2}{5}$ feet.

Pressure on fulcrum $= 20$ lbs.

2. A weight of 3 lbs. hung from one extremity of a straight lever balances a weight of 18 lbs. hung from the other extremity; find the ratio of the arms. *Ans.* 6 : 1.

3. In which of the three kinds of levers does the effort act at a mechanical disadvantage?—(*The Previous.*) *Ans.* The third kind.

4. What is meant by the efficiency of a machine?

Show that the efficiency of a lever when the weight acts at its middle point is the same as that of a lever whose fulcrum is between the power and the weight, and twice as far from the power as the weight.—(*The Previous.*)

5. Find the conditions of equilibrium in the case of a lever acted on by any two forces.—(*The Previous.*) *Ans.* $Pp = Qq$ (Fig. 42.)

6. Find the length of a lever of the second kind such that a power of 5 lbs. may support a weight of 12 lbs., and that their points of application may be a foot apart.—(*The Previous.*) *Ans.* $1\frac{1}{7}$ foot.

7. Define the mechanical advantage of a machine.

Find the position of the fulcrum in a lever of the first kind, length 1 foot, that weights of 3 lbs. and 7 lbs. may balance when hung at the ends.

Ans. $\frac{7}{10}$ of a foot from the end where the 3 lbs. weight is.

The following questions are taken from Woolwich Papers:—

8. Define a lever, and give instances of different kinds of levers. Describe the action of the muscles of the arm when it is held out horizontally from the elbow, and a weight is held in the hand.

9. State and prove the condition of equilibrium of a straight lever without weight acted on by any two forces at its extremities.

AB a uniform rod of weight (W), movable about a hinge at A, is sustained in equilibrium by a weight (P), attached to a string BCP passing over a smooth peg C, AC being vertical; if $\cos ACB = \frac{P}{W}$, show that $AC = AB$.

10. What is meant by mechanical advantage being gained or lost in the application of a lever? Give two examples in reference to contrivances in common use where mechanical advantage is lost by a lever. Two weights of 8 oz. and 4 oz. are in equilibrium at the opposite ends of a straight lever without weight; if 2 oz. be added to the greater weight the fulcrum must be moved through $\frac{1}{4}$ of an inch for equilibrium; find the length of the lever.

Ans. 1 foot.

11. A straight lever is acted upon by forces in the ratio of $\sqrt{3}+1 : \sqrt{3}-1$ at its extremities, and inclined at angles 60° and 30° to its length. Find the magnitude of the pressure on the fulcrum, and the direction in which it acts.

Ans. $2P\sqrt{2}$, and acts at an angle of 75° with lever.

12. A straight lever without weight is suspended by a string, and sustains in equilibrium two weights attached to the opposite extremities of the lever; investigate all the conditions of equilibrium. Is it necessary for equilibrium that the lever be horizontal?

Ans. (1) C. G. of weights beneath point of suspension. (2) No.

13. The pressure on the fulcrum of two weights in equilibrium, when suspended at the extremities of a straight lever 12 inches long, is 20 lbs., and the ratio of the distances of the fulcrum from each extremity of the lever is 3 : 2; find the weights. *Ans.* 12 and 8 lbs.

BALANCES.

156. **Definition of a Balance.**—A balance is a lever of the first kind with a scale-pan suspended from the extremity of either one or each of its arms.

157. **The Common Balance**

consists of a lever of the first kind, having equal arms from the ends of which scale-pans are suspended in such a way that the whole exactly balances on the fulcrum.

The Beam of the balance is a stiff piece of metal not represented in the figure (Fig. 43, p. 85) which, in well-constructed instruments, turns

about the edge M of a hard steel wedge (*called a knife edge*) which rests on a plate of smooth agate. When the beam is in a horizontal position its C. G. (viz. G) should be beneath the fulcrum M.

The scale-pans are hung from the beam by similar knife edges to that on which the beam turns.

158. **Requisites of a Good Balance.**—A good balance ought to be—1°. true; 2°. sensible; 3°. stable.

1°. A common balance is *true* if, when loaded with equal weights, the beam rests in a horizontal position.

2°. It is *sensible* if the smallest difference in the weights of the contents of the scales makes the angle of inclination of the beam to the horizon distinctly perceptible (or *sensible*).

3°. It is *stable* if when disturbed from a position of rest it readily and rapidly returns to that position.

159. **Requisites of a Good Balance, how Secured.** Let P and $P+p$ (Fig. 43) be the whole weights (including weights of

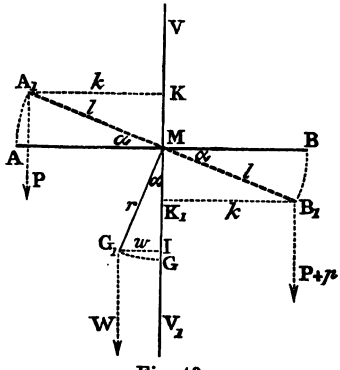

Fig. 43.

scale-pans, &c.) acting at A_1 and B_1, respectively. Then, if we make *the two arms of equal length, weight, and sectional area*, we have, by taking moments about M,

$$Pk + Ww = (P+p)k;$$

$$\therefore \quad Ww = pk;$$

$$\therefore \quad Wr\sin\alpha = pl\cos\alpha;$$

$$\therefore \quad \tan\alpha = \frac{pl}{Wr},$$

where W = weight of the *beam*, and G = C. G. of the beam. Hence—

1°. To be *true*, a common balance must have equal and similar arms. For, then, $\alpha = 0$, when $p = 0$, *i.e.* the beam is horizontal when the weights at its ends are equal. *If the arms be unequal the common balance is not true.* For in this case let $MB_1 = l_1$. Then, taking moments about M, we obtain

$$\tan \alpha = \frac{Pl_1 - Pl + pl_1}{W.r},$$

which does not vanish when $p = 0$, and consequently in this case the balance is not true.

2°. *Sensibility* is obtained by making the arms (l) long, and W and r, small. This implies that the beam should be long and light, and its C. G. *near* the point of suspension, but not so as to coincide with it, for then the beam would rest indifferently in any position.

3°. *Stability* is secured by making the arm l small in comparison with r, a condition at variance with the conditions of sensibility (2°). This condition of stability appears most properly by determining the time of oscillation of the beam when disturbed, and is a question of rigid dynamics.* In an elementary way we may say that stability depends on the magnitude of the moment which tends to restore equilibrium after the disturbance of the beam in a vertical plane, viz. Wr. If this be large the stability is great. This requires W and r to be both large, conditions inconsistent with sensibility (2°).

A compromise between sensibility and stability may be effected by making the arms long, and the weight of the beam small (yet not so small as to unduly lessen rigidity), and by making r as long as possible without seriously lessening sensibility.

160. General Remarks.

Sensibility is a more important requisite of a balance than stability, since the latter can be judged of by the eye by observing whether the oscillations of the beam on each side of the vertical are equal.

For Scientific Purposes, in which great accuracy is demanded, *sensibility* is the most important requisite; *for ordinary purposes, stability.*

---

* The time of oscillation is given by

$$t = \pi \sqrt{\frac{Mk^2 + 2P.l^2}{Mrg}},$$

where M = mass of beam, and k = beam's radius of gyration.

In our discussion of the balance we have considered the three points of suspension of the beam, and the two scale-pans to be in one straight line. On this supposition P does not enter the value of tan α, because the resultant of the two forces (PP) passes through M, and is neutralized by the reaction of the fulcrum at M.

EXERCISES.

1. What are the requisites of a good balance?

In a false balance, the arms being of unequal length, a weight is measured in one scale by P pounds, and in the other by Q pounds. Show that the lengths of the arms are to one another as $\sqrt{P} : \sqrt{Q}$.—(*The Previous.*)

2. How would you find the correct weight of a substance by means of a false balance?—(*The Previous.*) (*Vide* Ex. 11.)

3. If the arms of a false balance are 2 ft. and 2 ft. 1 inch, respectively, what is the true weight of a body which appears to weigh 10 lbs. when placed in the scale at the end of the shorter arm? *Ans.* $10\frac{5}{11}$ lbs.

4. When one of the scales of a common balance is loaded, a body appears to weigh W pounds when placed in one scale, and W' pounds when placed in the other. Prove that its true weight is an arithmetic mean of W and W'.

If the apparent weights of the body be 18 oz. and 20 oz., find the weight with which one of the scales was loaded.—(*The Previous.*) *Ans.* 1 oz.

5. If the beam of a false balance is uniform and heavy, show that the arms are proportional to the differences between the true and apparent weights.—(*The Previous.*)

6. Show that a balance is (other things being equal) more sensitive as its arms are longer.—(*The Previous.*)

7. Show that a balance is (other things being equal) more sensitive as its own weight is smaller.—(*The Previous.*)

8. If the arms of a false balance be without weight, and one arm longer than the other by one-ninth part of the shorter arm, and if in using it the substance to be weighed is put as often into one scale as the other; show that the seller loses five-ninths per cent. on his transactions.—(*The Previous.*)

9. For what purposes is sensibility the primary requisite of a good balance? Explain how it may be secured without sacrificing other advantages.—(*The Previous.*)

10. For what purposes is stability the primary requisite of a good balance? Explain how it may be secured without sacrificing other advantages.—(*The Previous.*)

11. **True Weight determined by a False Balance.**

Show that if W_1 and W_2 be the *apparent* weights of a body, as determined by placing it respectively in the two scales of a balance with unequal arms, that the *true* weight $W = \sqrt{W_1 W_2}$.

12. The lengths of the arms of a false balance are (a) and (b), and the weight W appears to balance P at the shorter arm (b), and Q at the longer arm (a). Show that if the balance be of uniform density and thickness,

$$\frac{a}{b} = \frac{P - W}{W - Q}.$$—(*Woolwich.*)

13. Find the position of equilibrium of a balance when the weights in each scale-pan, the lengths of the arms, the weights of the beam and of the scale-pans, and the position of the centre of gravity are given.—(*Woolwich.*)

161. The Common or Roman Steelyard.

The common steelyard is a lever of the first kind with very unequal arms, the longer arm being graduated, and provided with a movable weight P (Fig. 44).

The weight of any body suspended from the shorter arm is indicated by the point on the longer arm at which P must be placed to balance it.

162. To Graduate the Common or Roman Steelyard.

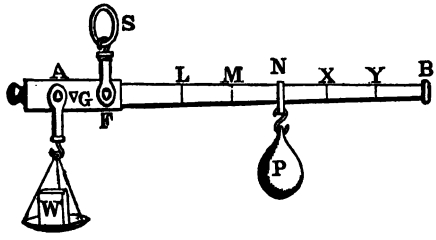

Fig. 44.

Let G = C. G. of the instrument, including the beam and scale-pan.

Let P at L balance instrument itself,

,, M ,, 1 lb. placed in scale,

,, N ,, 2 lb. ,,

Then, if successive divisions be marked along the arm each = LM, they will be the positions of P which correspond to successive additions of 1 lb. to the weight in the scale-pan.

For W being the weight of the machine,

$$P \cdot FL = W \cdot GF, \tag{1}$$

$$P(FL + LM) = W \cdot GF + 1 \times AF. \tag{2}$$

Subtract (1) from (2) and $P \cdot LM = AF.$ (3)

If n lbs. be balanced by P at N, then

$$P \cdot (FL + LN) = W \cdot GF + n \cdot AF;$$

$$\therefore \quad P \cdot FL + P \cdot LN = W \cdot GF + n \cdot AF.$$

But $P \cdot FL = W \cdot GF$; by (1)

$$\therefore \quad P \cdot LN = n \cdot AF = nP \cdot LM; \text{ by (3)}$$

$$\therefore \quad LN = nLM.$$

Let $n = 2$, then $LN = 2LM.$

,, $= 3$, then $LN = 3LM$, which gives N at X,

,, $= 4$, then $LN = 4LM$, ,, ,, Y,

and so on

In the graduation of the steelyard here considered we have assumed G to be in the shorter arm. Should G be either *at* the fulcrum F, or *in the longer arm*, the corresponding graduation can be effected by analogous methods which are left as exercises to the student.

163. The Sensibility of the Common or Roman Steelyard

is greater the greater the distance between the two points of suspension of P necessary to balance two weights of given difference. Hence the sensibility is increased *by increasing* FA (Fig. 44, p. 88), which increases the momental effect of a given weight in the scale-pan, or *by diminishing* P, which necessitates a longer arm to give the same momental effect as that produced by a larger movable weight.

164. The Danish Steelyard.

The Danish steelyard consists of a straight bar HA, having a heavy knob H at one end, and a scale-pan at the other.

The fulcrum F is movable, and the weight of any body is determined by the position of F. Let P be the weight of the steelyard (including the scale-pan), and let it act at G.

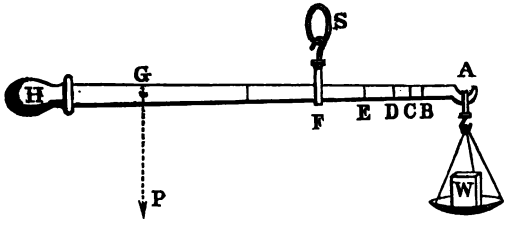

Fig. 45.

Graduation.—In all cases

$$P.GF = W.AF.$$

Let $W = P$, then $GF = AF$; $\therefore AF = \frac{1}{2}AG$,

,, $W = 2P$, then $GF = 2AF$; $\therefore AF = \frac{1}{3}AG$,

,, $W = 3P$, then $GF = 3AF$; $\therefore AF = \frac{1}{4}AG$,

and thus the graduation is effected, the points E, D, C, &c., representing different positions of F.

A very common application of the principle of the steelyard can be seen in the weighing-machines employed at most railway stations.

EXERCISES.

1. Explain the construction and graduation of the common steelyard.

A uniform rod being divided into twenty equal parts by graduations, the fulcrum is placed at the first graduation. The greatest and least weights which the instrument can weigh are 2 lbs. and 20 lbs. Find its weight.—(*The Previous.*) *Ans.* $\frac{1}{2}$ lb.

2. Explain the construction and graduation of the Danish steelyard.

The distance between the zero of graduation, and the end of the instrument is divided into twenty equal parts, and the greatest weight which can be weighed is 3 lbs. 9 oz. Find the weight of the instrument.—(*The Previous.*) *Ans.* 3 oz.

3. Show that the sensitiveness of the common steelyard is independent of the magnitude of the movable weight.

A steelyard is 12 inches long, and with the scale-pan weighs 1 lb., the C. G. of the two being 2 inches from the end to which the scale-pan is attached. Find the position of the fulcrum when the movable weight is 1 lb., and the greatest weight that can be ascertained by means of the steelyard is 12 lbs.—(*The Previous.*) *Ans.* 1 inch from scale-pan end.

4. Define the sensitiveness of a balance, and show that in the Danish steelyard the sensitiveness diminishes as the weight which is ascertained by means of it increases. Find the length of a Danish steelyard weighing 1 lb., when the distance between the graduations 4 lbs. and 5 lbs. is 1 inch.—(*The Previous.*) *Ans.* $3\frac{1}{2}$ feet from C. G. of instrument to graduated end.

5. If the bar in the Danish steelyard rest with the fulcrum half way between the first and second graduations, show that the weight in the scale is $\frac{1}{7}$ of the weight of the bar.—(*The Previous.*)

6. Where must the C. G. of the Roman steelyard be situated in order that any movable weight whatever may be used with it?

Ans. At the fulcrum.

7. In a common steelyard the length of the rod is 2 feet, its weight 2 lbs., the distance of its C. G. from the fulcrum 1 inch towards the end of the shorter arm, the distance of the point where the weight is suspended from the fulcrum 2 inches, and the movable weight 6 ozs. Find the greatest weight which can be weighed.—(*Woolwich.*) *Ans.* $3\frac{1}{2}$ lbs.

8. In the Danish steelyard show that if the successive values of W form an A. P., the distances of the corresponding graduations will form a harmonical series.

9. AB is a Roman steelyard 10 feet long; its C. G. is 9 inches from A, and the fulcrum 6 inches from A. If the weight of the steelyard be 8 lbs., and that of the movable weight 1 lb.; find how many inches from B the graduation marked 12 lbs. will be. *Ans.* 66 inches.

10. A common steelyard supposed uniform is 40 inches long, the weight of the beam is equal to the movable weight, and the greatest weight that can be weighed by it is four times the movable weight; find the place of the fulcrum.—(*Woolwich.*)

Ans. 10 inches from middle of beam on the side remote from movable weight.

165. The Wheel and Axle.

The wheel and axle consists of two cylinders (A and B) of different radii, the larger (A) being called the wheel, and the smaller (B) the axle. They are rigidly connected, and revolve on two cylindrical pivots (i and i') (called journals), which rotate in fixed bearings. The geometrical axes of the cylinders and pivots form one horizontal straight line.

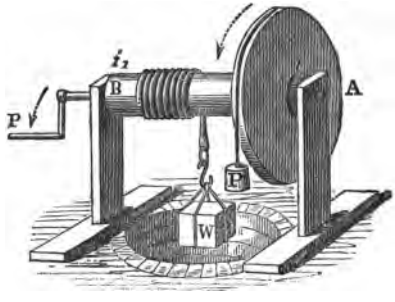

Fig. 46.

A rope having one end attached to the axle sustains at the other end the weight or resistance (W). The effort (P) is applied by another rope attached to the wheel, and wrapped round it in a direction opposite to that of the rope on the axle.

166. Equilibrium of the Wheel and Axle.

We may suppose P and W to act in the same plane perpendicular to the axis of rotation.

Let the adjoining figure represent a section of the wheel and axle in this plane, G being the end of the axis of rotation.

We may regard the machine as made up of an indefinite number of levers having a common fulcrum G, and coming successively into play. Taking moments about G, we have

$$P \times R = W \times r,$$

i. e. effort × radius of wheel = resistance × radius of axle. Hence, also,

$$P \times C = W \times c$$

(C and c being circumferences of wheel and axle, respectively).

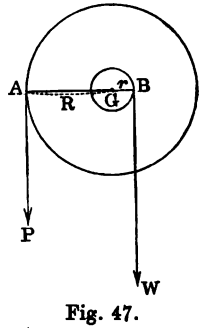

Fig. 47.

167. **Mechanical Advantage of the Wheel and Axle.**

$$= \frac{W}{P} = \frac{C}{c},$$

which may be increased either by making the wheel larger or the axle smaller, or by doing both. Undue enlargement of the wheel makes the machine *unwieldy*, while the diminution of the thickness of the axle produces *weakness*. In the differential wheel and axle we can obtain mechanical advantage without these accompanying defects (*vide* Art. 169, end).

168. **The Winch, Windlass, and Capstan**

are instances of the wheel and axle in which the effort is differently applied, as shown in the adjoining diagram.

The **Winch**

The **Windlass**

The **Capstan**

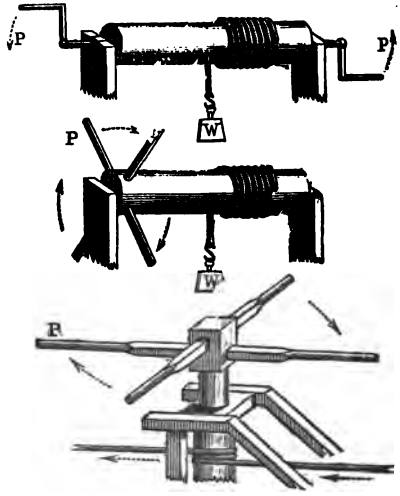

Fig. 48.

In the windlass and capstan the handles are rigidly attached to the axle, and are perpendicular thereto. In the capstan the axle is vertical.

169. **The Differential Wheel and Axle.**

The diagram (Fig. 49) explains itself. When the handles are turned so as to lift the weight, one branch of the rope attached to W winds round the larger cylinder c, and the other unwinds from c_1. By a single revolution of

the cylinders, the unwound portion of the rope is therefore shortened by a length $= c - c_1$; therefore W is raised a distance $= \frac{c - c_1}{2}$. Hence, by the principle of work done* (Part II., Chap. V.), $P \times C = W.\frac{c - c_1}{2}$,

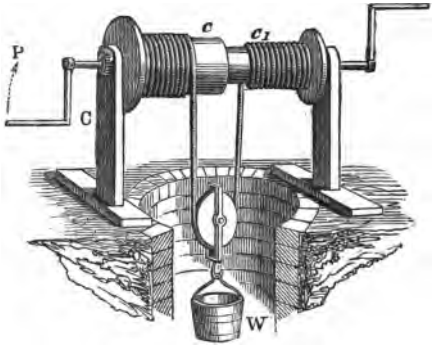

Fig. 49.

where C is the circumference of the circle described by P, and c and c_1 are the circumferences of the two cylinders on which the rope is wound.

The same result may be obtained by taking moments about the common axis of rotation of the cylinders. Thus

It is evident that the tensions on each branch of the cord $= \frac{W}{2}$, since these two tensions together support W; therefore if

R = radius of circle described by P,

r = radius of the cylinder c,

r_1 = radius of the cylinder c_1,

we obtain, by taking moments about the central axis,

$$P.R + \tfrac{1}{2}Wr_1 = \tfrac{1}{2}Wr;$$

$$\therefore \quad P.R = \tfrac{1}{2}W(r - r_1); \quad \therefore \quad P.C = W\frac{c - c_1}{2}.$$

Mechanical Advantage $= \frac{W}{P} = \frac{2C}{c - c_1}$, which may be increased at pleasure by making the radii of the cylinders c and c_1 more and more nearly equal. This can evidently be done without making the machine either *unwieldy* or *weak* (*vide* Art. 165).

* This mode of solution is adopted on account of its simplicity.

Exercises on the Wheel and Axle.

1. A wheel and axle is used to raise a bucket from a well; the radius of the wheel is 2 feet, and that of the axle 3 inches. Find the force which must be employed at the circumference of the wheel to *just* balance the bucket and its contents, supposed to weigh 8 cwt. *Ans.* 1 cwt.

2. If the radius of the wheel be 3 feet, and that of the axle 6 inches, what force will be required to balance a resistance of 1 ton? *Ans.* 373½ lbs.

3. The circumference of the wheel being 12 feet, determine that of the axle when a force of 24 lbs. applied at the circumference of the wheel balances a resistance of 192 lbs. hanging from the axle. *Ans.* 18 inches.

4. In a windlass the radius of the axle is 6 inches, and the length of the handle is 3 feet; determine what resistance can be just overcome by an effort of 1 cwt. *Ans.* 6 cwt.

5. Twelve sailors, each exerting a force of 56 lbs., work a capstan whose levers are 8 feet long; the radius of the axle is 18 inches: what resistance can they unitedly sustain? *Ans.* 384 cwt.

6. Two men able to exert forces of 168 lbs. and 224 lbs., respectively, work the handles of a winch. The radius of the axle is 7 inches: what must be the length of the arm of the winch that the men may be just able to raise a weight of 1 ton? *Ans.* 3½ feet.

7. In the differential wheel and axle the radii of the two parts of the axle being 6 and 8 inches, respectively, what resistance will a force of 56 lbs. applied to a handle 3 feet long overcome? *Ans.* 36 cwt.

8. Define the wheel and axle, and find the ratio of the power to the weight when they are in equilibrium.

Four wheels and axles, in each of which the radii are in the ratio of 5 to 1, are arranged so that the circumference of each axle is applied to the circumference of the next wheel. What power will balance a weight of 1875 lbs.?—(*The Previous.*) *Ans.* 3 lbs.

9. Four sailors, each exerting a force capable of raising 116 lbs., raise an anchor by means of a capstan, whose radius is 1 foot 2 inches, and whose spokes are 8 feet long (measured from the axis). Find the total force they exert.—(*The Previous.*) *Ans.* 4253½ lbs.

170. Toothed Wheels.

Toothed wheels are circular discs of wood or metal whose circumferences are cut into equal teeth at equal distances apart all round. If two such wheels (*A* and *B*, Fig. 50), whose teeth are of equal size, be arranged to rotate on parallel axes, in such a way that a tooth of one fits between two contiguous teeth of the other, a rotation given to one will impart an opposite rotation to the other. The wheel communicating the motion to the other is called the *driver*, and the driven wheel the *follower*.

If the teeth of the wheels be large, the mutual pressure between them varies during the motion, both in magnitude and direction, unless the teeth be of peculiar construction. A general discussion of this important subject is beyond the limits of the present work. If, however, the teeth be small in comparison with the sizes of the wheels, we may consider the mutual pressure between the wheels constant both in magnitude and direction, in which case we can treat the subject by elementary methods.

171. Ratio of Effort to Resistance in Toothed Wheels, when the Teeth are small.

Let *A* and *B* represent two toothed wheels whose teeth are small, and whose radii are p and q, respectively, and let them revolve on equal axles when radii are each $= r$.

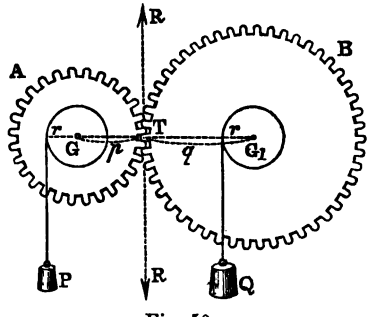

Fig. 50.

The mutual pressure (R) between the wheels acts along the common tangent, as shown.

Then, from *A*, $Pr = Rp$, (1)

,, *B*, $Qr = Rq$. (2)

Dividing (1) by (2), $\frac{P}{Q} = \frac{p}{q}$;

$$\therefore \quad \frac{P}{Q} = \frac{\text{circumference of } A}{\text{circumference of } B};$$

$$\therefore \quad \frac{P}{Q} = \frac{\text{no. of teeth in } P\text{'s wheel}}{\text{no. of teeth in } Q\text{'s wheel}}$$

the teeth in each wheel being of the same size, and at the same distance apart.

172. Pinion—Leaves.

The smaller of two cogged wheels geared to work on one another is sometimes called a *pinion*, and its teeth *leaves*. This pinion frequently forms part of the axle of a larger toothed wheel, by the rotation of which *it* is also made to rotate and communicate motion to another toothed wheel.

The Rack and Pinion is a simple case of the application of a toothed wheel to mechanical purposes. It consists of a small cogged wheel whose teeth work into a vertical bar also fitted with teeth. By this arrangement motion round an axis is converted into motion in a straight line.

EXERCISES.

1. Two toothed wheels whose radii are a and b, respectively, rotate on equal horizontal axles in such a way that their teeth interlock. Flexible cords are attached to the axles round which they are wound in opposite directions, and sustain weights P and Q, respectively, which hang vertically. If the system balances, prove that

$$\frac{P}{Q} = \frac{\text{no. of teeth in } P\text{ s wheel}}{\text{no. of teeth in } Q\text{'s wheel}}$$

where the teeth are supposed to be small in comparison with the sizes of the wheels.

2. A wheel of 101 teeth drives another of 10 teeth, show that if the driver makes 20 revolutions per minute, the follower will make 202 revolutions per minute.

3. A, B, and C are three parallel axles: A carries a wheel of a teeth which works on a pinion attached to B containing β teeth: B also carries a wheel containing b teeth, which works on a pinion attached to C containing γ teeth; find the number of revolutions C makes per minute when A revolves n times per minute.

Ans. $\frac{a}{\beta} \cdot \frac{b}{\gamma} n$ revolutions per minute.

4. If in Ex. 3 there are $x + 1$ axles, and the drivers contain a teeth each, and the followers b teeth each: show that the number of revolutions per minute made by the last axle will be

$$n \cdot \left(\frac{a}{b}\right)^x.$$

NOTE.—On account of their importance separate chapters are devoted to the Inclined Plane and Pulley.

CHAPTER XIII.

II.—THE INCLINED PLANE.

173. We shall in this Chapter consider the equilibrium of a body (W) sustained on an inclined plane by a force (P), or forces acting upon it. We shall first consider the plane to be smooth, and the body (W) to be sustained thereon by a single force (P). It will be convenient to consider three cases.

Case I.

174. **When P acts Parallel to Inclined Plane.**

P, W, and R equilibrate;

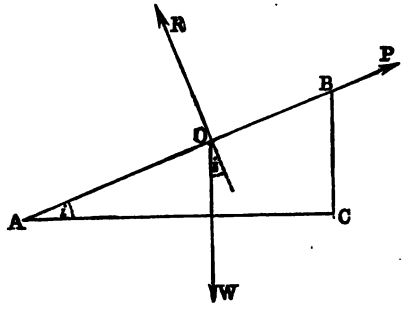

Fig. 51.

$\therefore$ (Art. 60) $\dfrac{P}{\sin \hat{WR}} = \dfrac{W}{\sin \hat{RP}} = \dfrac{R}{\sin \hat{PW}}$;

$\therefore$ $\dfrac{P}{\sin i} = \dfrac{W}{\sin 90°} = \dfrac{R}{\sin (90° - i)}$;

$\therefore$ $\dfrac{P}{\sin i} = \dfrac{W}{1} = \dfrac{R}{\cos i}$;

$\therefore$ $P = W \sin i$, (1)

$R = W \cos i$. (2)

EXERCISES.

1. A weight of 100 lbs. is sustained on a smooth inclined plane by a force parallel to the length of the plane. If the angle of inclination of the plane to the horizon be 30°, determine the magnitude of the force, and the resistance of the plane. *Ans.* $P = 50$ lbs, $R = 50\sqrt{3}$ lbs.

2. Determine P and R when $i = 60°$ in Ex. 1.
Ans. $P = 50\sqrt{3}$ lbs., $R = 50$ lbs.

3. Determine P and R when $i = 45°$ in Ex. 1.
Ans. $P = 70{\cdot}71$ lbs $= R$.

4. A truck is sustained on an inclined plane whose gradient* is 1 in 10 by a chain parallel to length of plane; determine the tension on the chain, and the pressure on the plane.
Ans. Tension = 2 tons; pressure = 19·9 tons.

5. What is the inclination of a smooth plane to the horizon (Art. 40) when a force acting parallel to the plane, which keeps a weight W in equilibrium on the plane, is equal to the pressure on the plane? *Ans.* 45°.

6. If the power acting along an inclined plane, the pressure on the plane, and the weight be in the ratio $1 : \sqrt{3} : 2$; find the inclination of the plane to the horizon.—(Mr. Panton.) *Ans.* 30°.

7. A weight of 20 lbs. is supported by a string fastened to a point in an inclined plane; the plane is gradually raised, and the string breaks when the inclination is 30°: find what weight hanging vertically the string can sustain.—(Dr. Traill.) *Ans.* 10 lbs.

8. Determine the inclination of a smooth inclined plane when a force of 6 lbs. parallel to it will *just* sustain a weight of 12 lbs. upon it. What is the amount of pressure on the plane?—(Mr. Bernard.)
Ans. $i = 30°$; $R = 6\sqrt{3}$ lbs.

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* The word *gradient* is here used to mean $\frac{\text{height}}{\text{length}}$. By engineers the word is used to mean $\frac{\text{height}}{\text{base}}$.

Vide Warren's *Table Book*, p. 42. (Longmans, Green, & Co., London.)

9. Two weights W and W' support each other on two inclined planes, making with the horizon angles $= i$ and i', respectively, by means of a flexible string passing over the common vertex of the planes. Find the ratio of W to W', the tension on the string, and the pressures R and R' on the planes.

Ans. $\dfrac{W}{W'} = \dfrac{\sin i'}{\sin i}$,

$T = W \sin i = W' . \sin i'$,

$R = W \cos i$, $R' = W' \cos i'$.

10. A railway train weighing 120 tons is drawn up an incline of 1 in 60 by means of a chain and a stationary engine. What tension at least must the chain be able to bear? N.B. $\sin i = \frac{1}{60}$. *Ans.* 2 tons.

Case II.

175. When P acts Parallel to Horizon* (*i. e.* to Base of Inclined Plane), *vide* Art. 40.

As before,

$$\frac{P}{\sin \widehat{WR}} = \frac{W}{\sin \widehat{RP}} = \frac{R}{\sin \widehat{PW}};$$

$$\therefore \frac{P}{\sin i} = \frac{W}{\sin (90° - i)} = \frac{R}{\sin 90°};$$

$$\therefore \frac{P}{\sin i} = \frac{W}{\cos i} = \frac{R}{1};$$

$$\therefore P = \frac{W \sin i}{\cos i} = W \tan i, \quad (1)$$

$$R = \frac{W}{\cos i} = W \sec i. \quad (2)$$

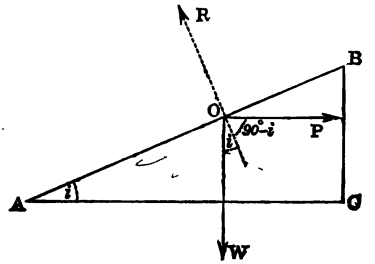

Fig. 52.

* The word horizon simply means a horizontal plane at any place (the ground for example, if horizontal). The word is popularly used to signify the boundary of vision, *i. e.* the intersection of a horizontal plane with the celestial vault.

EXERCISES.

1. What horizontal force is necessary to sustain a weight of 12 lbs. on an incline of 3 in 5?—(Dr. Traill.) *Ans.* 9 lbs.

2. If $W = 10$ tons, and $i = 45°$, find P and R in Case II.

Ans. $P = 10$ tons, $R = 14{\cdot}142$ tons. Observe R is $> W$.

3. In Case II. prove that $R^2 = W^2 + P^2$.

4. If $W = 80$ lbs., and the height of the inclined plane be to its base as 9 to 40, find P and R. *Ans.* $P = 18$ lbs.; $R = 82$ lbs.

5. If $W = 24$ lbs., and $P = 18$ lbs.; find R and i.

Ans. $R = 30$ lbs.; $i = \tan^{-1}\frac{3}{4}$.

6. Find the horizontal force necessary to support a weight of 1 lb. on a smooth plane which rises 3 in 5.—(Dr. Tarleton.) *Ans.* $\frac{3}{4}$ lb.

7. A weight W is placed on a smooth plane inclined at an angle i to the horizon; find the least force acting parallel to the horizon which will sustain it.—(Mr. Panton.) *Ans.* $W \tan i$.

8. What force parallel to the horizon will sustain a weight of 250 lbs. on a smooth plane inclined to the horizon at an angle of 60°?—(Mr. Panton.)

Ans. $250\sqrt{3}$ lbs.

Case III.

176. When P makes an Angle θ with the Inclined Plane.

As before,

$$\frac{P}{\sin \widehat{WR}} = \frac{W}{\sin \widehat{RP}} = \frac{R}{\sin \widehat{PW}}; \quad \therefore \quad \frac{P}{\sin i} = \frac{W}{\sin (90° - \theta)} = \frac{R}{\sin x};$$

$$\therefore \quad \frac{P}{\sin i} = \frac{W}{\cos \theta} = \frac{R}{\cos (i + \theta)}$$

[since $x = 90° - (i + \theta)$].

$$\therefore \quad P = \frac{W \sin i}{\cos \theta}, \quad (1)$$

$$R = \frac{W \cos (i + \theta)}{\cos \theta} \quad (2)$$

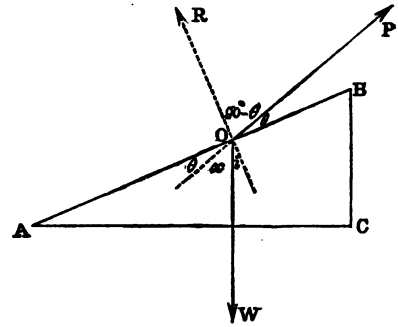

Fig. 53.

EXERCISES.

1. In the inclined plane if the force make an angle θ with the inclined plane; prove that

$$P\cos\theta = W\sin i.$$—(Mr. M'Cay.)

2. If $W = 100$ lbs; $i = 30°$, and $\theta = 45°$; find P and R.

Ans. $P = \frac{100}{\sqrt{2}}$ lbs.; $R = 50(\sqrt{3}-1)$ lbs.

3. If $i = 60°$ and $R = \frac{W}{2}$; find θ and P. *Ans.* $\theta = 0$, and $P = \frac{W\sqrt{3}}{2}$.

4. If $i = 60°$ and $R = W$; find θ and R.

Ans. $\theta = 60°$, and $R = 0$.

N.B.—In this exercise P acts vertically, and sustains W without the inclined plane.

5. For a given inclination i, and a given weight W, determine the direction in which P should act so that W may be supported on the plane with a minimum expenditure of force. *Ans.* P should act along the plane.

6. For a given value of P, and a given value of W, show that there are two directions in which P may act so as to support W on a smooth plane of given inclination.

7. For given values of i and W, find how P must act so that R may be a minimum. *Ans.* $\theta = 90° - i$.

8. Find the direction in which a force of 8 lbs. must act on a weight of 12 lbs., placed on a smooth inclined plane whose inclination to the horizon is 30°, to support the weight.

Ans. It must make with inclined plane an angle θ such that $\cos\theta = \frac{2}{3}$.

9. Find the magnitude of the least possible force which can support a weight of 15 lbs. on a smooth plane inclined at an angle of 30° to the horizon.—(*The Previous.*)

Ans. $\frac{10}{\sqrt{3}}$ lbs.

10. What is the inclination to the horizon of the steepest plane on which a power of 5 lbs. will support a weight of 10 lbs. ?—(*The Previous.*)

Ans. 30°.

11. If $P = R$, find θ (Case III.) for a given inclination (i).

Ans. $\theta = 90° - 2i$.

12. If the weight, power, and pressure on a smooth inclined plane are in the ratio $\sqrt{3} : 1 : 1$, find the inclination of the plane, and the direction of the force. *Ans.* Plane makes 30° with horizon, and force 30° with plane.

177. Inclined Plane by Resolution.

All the results of Arts. 174, 175, and 176 may be obtained by resolving the forces involved along suitable directions.

Case I.

In Case I., resolving along and perpendicular to plane, we obtain

$$P - W\cos(90° - i) = 0\,;\ \text{whence } P = W\sin i, \qquad (1)$$

and

$$R - W\cos i = 0\,;\ \text{whence } R = W\cos i. \qquad (2)$$

Case II.

Here, resolving along plane,

$$P\cos i - W\sin i = 0\,;\ \text{whence } P = W\tan i. \qquad (1)$$

Resolving perpendicular to P,

$$R\cos i - W = 0\,;\ \text{whence } R = W\sec i. \qquad (2)$$

Case III.

Resolving along plane,

$$P\cos\theta - W\cos(90° - i) = 0\,;\ \text{whence } P = \frac{W\sin i}{\cos\theta}. \qquad (1)$$

Resolving perpendicular to P,

$$R\cos\theta - W\cos(i + \theta) = 0\,;\ \text{whence } R = \frac{W\cos(i + \theta)}{\cos\theta}. \qquad (2)$$

We have resolved in all cases here perpendicular to P and R, respectively. By resolving along, and perpendicular to, the inclined plane, we obtain the same results, or by resolving along any two suitable directions.

178. Inclined Plane with Friction.

In considering the inclined plane with friction it will be sufficient to consider the general case (Case III.) when P makes an angle θ with the inclined plane, from which the other two cases may be deduced by assigning proper values to θ. We have to consider two limiting cases—

1°. When W is about to *slip up* the plane, in which case let

$P = P_1$, and $R = R_1$.

2°. When W is about to *slip down* the plane, in which case let

$P = P_2$, and $R = R_2$.

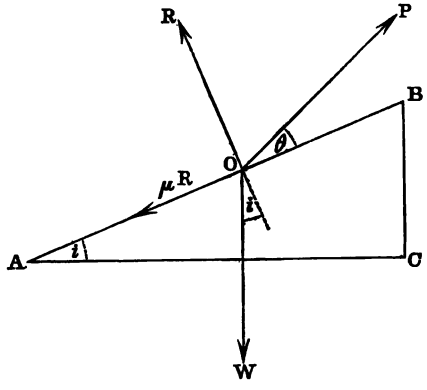

Fig. 54.

1°. W about to slip *up*.

Here the normal resistance being R_1, the friction developed in the state bordering on motion is μR_1, and acts *down* the plane. By resolving *along*, and perpendicular to plane, we obtain

$$P_1 \cos\theta - \mu R_1 - W \sin i = 0,$$

and

$$R + P_1 \sin\theta - W \cos i = 0. \quad \text{Whence}$$

$$P_1 = \frac{W(\sin i + \mu \cos i)}{\cos\theta + \mu \sin\theta} = \frac{W \sin(i+\lambda)}{\cos(\theta - \lambda)}, \tag{1}$$

$$R_1 = \frac{W(\cos i \cos\theta - \sin i \sin\theta)}{\cos\theta + \mu \sin\theta} = \frac{W \cos\lambda \cdot \cos(i+\theta)}{\cos(\theta - \lambda)}, \tag{2}$$

where λ = angle of friction = $\tan^{-1}\mu$.

2°. W about to slip *down*.

Here μR_2 acts *up* the plane, and, as before,

$$P_2 = \frac{W(\sin i - \mu \cos i)}{\cos\theta - \mu \sin\theta} = \frac{W \sin(i-\lambda)}{\cos(\theta + \lambda)}, \tag{1}$$

$$R_2 = \frac{W(\cos i \cos\theta - \sin i \sin\theta)}{\cos\theta - \mu \sin\theta} = \frac{W \cos\lambda \cdot \cos(i+\theta)}{\cos(\theta + \lambda)}. \tag{2}$$

If P have any value between P_1 and P_2, the weight W will remain in equilibrium on the inclined plane.

EXERCISES.

1. A body weighing 100 lbs. is placed on a rough inclined plane whose angle of inclination to the horizon is 60°. Find the minimum force which, acting on the body parallel to the inclined plane, will maintain it in equilibrium, the coefficient of friction between the body and plane being $\frac{1}{\sqrt{3}}$.

When the force maintaining equilibrium is a minimum, the body is on the point of slipping down the plane, and therefore μR acts upwards. Resolving in general terms *along*, and *perpendicular* to plane, we obtain

1°. $$P + \mu R - W \sin i = 0.$$

2°. $$R - W \cos i = 0.$$

Whence $$P = W(\sin i - \mu \cos i)$$

$$= 100\left(\sin 60° - \frac{1}{\sqrt{3}} \cos 60°\right) \text{ in this exercise}$$

$$= 100\left(\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{3}} \cdot \frac{1}{2}\right) = 57{\cdot}7 \text{ lbs.}$$

2. In Ex. 1 find the maximum value of P consistent with equilibrium. In this case the body is about to slip up the plane, and

$$P = W(\sin i + \mu \cos i)$$

$$= \frac{100 \times 2}{\sqrt{3}} = 115{\cdot}4 \text{ lbs.}$$

3. A body of 1 lb. is placed on an inclined plane, and is acted on by a horizontal force of 4 lbs. If the body be on the point of slipping up the plane under these circumstances, determine the angle of friction when $\tan i = \frac{3}{4}$, where i = inclination of the plane to the horizon.

Resolving along the plane,

$$\mu R + W \sin i = P \cos i. \qquad (1)$$

Resolving perpendicular to plane,

$$R = W\cos i + P\sin i. \qquad (2)$$

But $\tan i = \frac{3}{5}$;

$$\therefore \quad \sin i = \frac{3}{\sqrt{34}},^*$$

and $$\cos i = \frac{5}{\sqrt{34}};^*$$

$$\therefore \text{ from (2) } R = \frac{1\times 5}{\sqrt{34}} + \frac{4\times 3}{\sqrt{34}} = \frac{17}{\sqrt{34}}.$$

Putting this value for R in (1), we obtain

$$\mu\frac{17}{\sqrt{34}} + \frac{3}{\sqrt{34}} = \frac{20}{\sqrt{34}};$$

$$\therefore \quad 17\mu = -3 + 20;$$

$$\therefore \quad \mu = \frac{17}{17} = 1;$$

$$\therefore \quad \tan\lambda = 1;$$

$$\therefore \quad \lambda = 45°.$$

4. The height, base, and length of a rough inclined plane are to one another as 3 : 4 : 5. Find what force must be exerted parallel to the plane to sustain on it a weight of 20 lbs.—(Mr. Purser.)

Ans. $P = (12 - 16\mu)$ lbs.

5. A weight is placed on a plane inclined to the horizon at an angle of 30°. If the angle of friction be 60°, compare the forces required to move the weight up and down, these forces being applied in both cases parallel to the plane. —(Mr. Purser.)

Ans. $P = 2W$ (up).

$P = -W$ (down).

---

* *Vide* Warren's *Plane Trigonometry*, Exercise 6, p. 31, third edition. (Longmans, Green, & Co., London.)

6. The gradient of a smooth inclined plane being 5 in 13; determine the horizontal force sufficient to maintain a weight of 52 lbs. on it. If the plane be rough, a horizontal force of half this magnitude is found sufficient; determine the coefficient of friction.—(Mr. Purser.)

Ans. 20 lbs.; $\mu = \frac{5}{12}$.

7. Determine the least horizontal force which will maintain in equilibrium a body of weight W on a rough inclined plane whose inclination to the horizon is 60°, being given the coefficient of friction = 1.

Ans. $P = W\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$.

8. A body is supported on a rough inclined plane by a force acting along the plane. If the greatest magnitude of the force when the plane is inclined at the angle α to the horizon is equal to the least value of the force when the plane is inclined at the angle α' to the horizon; show that

$$\mu = \frac{\sin\alpha \sim \sin\alpha'}{\cos\alpha + \cos\alpha'}.$$—(*Woolwich.*)

9. A heavy body rests on a rough inclined plane; if $\tan\beta$ be the coefficient of friction, and α the inclination of the plane; prove that the greatest and least force acting upwards parallel to the plane which will support the weight are in the ratio of

$$\sin(\alpha+\beta) \text{ to } \sin(\alpha-\beta).$$—(*Woolwich.*)

10. A force P acts parallel to a rough inclined plane, and is just on the point of moving a weight W up the plane; if α be the inclination of the plane, and if the coefficient of friction be $\dfrac{\sin\left(45°-\frac{\alpha}{2}\right)}{\sin\left(45°+\frac{\alpha}{2}\right)}$, examine whether any mechanical advantage is obtained by the inclined plane. *Ans.* No.—(*Woolwich.*)

11. A plane of small slope rises one foot vertical for n feet horizontal, and the coefficient of friction is $\frac{1}{m}$. If a weight W be placed on the plane, show that the force P which will *just* move W up the plane is nearly

$$= W\left(\frac{1}{m}+\frac{1}{n}\right).$$—(*Woolwich.*)

12. A body rests on a rough inclined plane; show how to determine the inclination of the plane when the body is on the point of slipping, assuming the coefficient of friction = $\frac{1}{2}$.—(Mr. Williamson.)

Ans. The body will slip when the inclination of the plane to the horizon = $\tan^{-1}\frac{1}{2}$, *i. e.* the angle of friction.

General Exercises on the Inclined Plane.

13. A sphere of weight W is placed on two smooth inclined planes whose line of intersection is horizontal; determine the pressures on the planes.—(*Degree*, T. C. D., 1888.)

Ans. $R_1 = \frac{W \sin \alpha_2}{\sin(\alpha_1 + \alpha_2)}$; $R_2 = \frac{W \sin \alpha_1}{\sin(\alpha_1 + \alpha_2)}$, where α_1 and α_2 are the inclinations of the planes to the horizon.

14. What weight can be supported on a smooth plane, inclined at an angle of 45° to the plane of the horizon, by a horizontal force of 3 lbs., and a force of 4 lbs. parallel to the inclined plane, acting together, and situated in the same vertical plane, perpendicular to the intersection of the inclined plane and the plane of the horizon?—(*Intermediate*, 1884.)

Ans. $\frac{3 + 4\sqrt{2}}{\sqrt{2}}$ lbs.

15. On a smooth inclined plane, whose inclination to the horizon is i, a weight W is placed, and supported by three strings making angles A, B, and C, with the inclined plane. The strings pass over smooth pulleys, and have weights of 9, 8, and 10 lbs. attached to them. If $\sin i = \frac{1}{10}$, $\cos A = \frac{1}{3}$, $\cos B = \frac{1}{4}$, and $\cos C = \frac{1}{5}$; find the magnitude of W in order that there should be equilibrium.—(Dr. Tarleton, *J.S.*) *Ans.* 140 lbs.

16. A heavy circular disc is kept at rest on a rough inclined plane by a string parallel to the plane and touching the circle. Show that the disc will slip on the plane if the coefficient of friction is less than $\frac{1}{2}\tan i$, where i = slope of plane.—(*Woolwich*, 1886.)

17. Explain how the use of the inclined plane may facilitate mechanical operations. A number of loaded trucks, each containing one ton, on one part of a tramway whose inclination to the horizon is α, supports an equal number of empty trucks on another part whose inclination is β. Find the weight of a truck.—(*Woolwich.*)

Ans. $\frac{\sin \alpha}{\sin \beta - \sin \alpha}$ tons.

18. A wedge with angle 60° is placed upon a smooth table, and a weight of 20 lbs. on the slant face is supported by a string lying on that face, passing through a smooth ring at the top, and supporting a weight W hanging vertically. Find the magnitude of W. Find also the force necessary to keep the wedge at rest—(1) when the ring is not attached to the wedge; (2) when it is so attached.

Ans. $W = 10\sqrt{3}$ lbs.; (1) $5\sqrt{3}$ lbs.; (2) No force.

179. The Screw.

The screw is a modification of the inclined plane. It consists of an inclined plane wrapped round a right circular cylinder, in such a way as to form a uniform projecting thread *abcd*, traced round the surface, and making a constant angle i with the horizon. The cylinder fits into a block C, pierced with an equal cylindrical aperture, on the inner surface of which is cut a groove, the exact counterpart of the projecting thread *abcd*, and called *the nut*. When the block is fixed so as to have the aperture in a vertical position, the only manner in which the screw can move is up or down by rotation about a vertical axis.

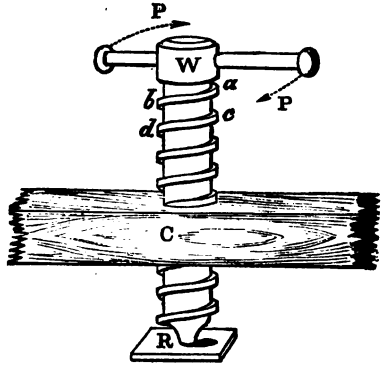

Fig. 55.

Let us suppose a weight W placed on the top of the screw, and let the screw be supposed to move *without friction* in the nut. The weight will descend by the revolution of the screw. If this motion be prevented by a *horizontal force* P applied at the circumference of the cylinder, $P = W \tan i$ (Case I., inclined plane), and $R = W \sec i$, where R = resistance perpendi-

cular to thread. The resistance here considered is supposed to act at some specific point of the inside of the nut at which we conceive P and W for the moment to act.

Since $$\tan i = \frac{\text{interval between threads}}{\text{circumference of cylinder}},$$

we have $$P = \frac{W \times \text{interval between threads}}{\text{circumference of cylinder}};$$

therefore effort (P) × circumference of cylinder = resistance (W) × interval between threads.

180. **Equilibrium of the Screw by the Principle of Work Done.** (*Vide* Part II., Chap. V.)

In many applications of the screw the effort (P) is applied by means of a handle, or handles attached to the cylinder. In this, and in all other cases, the equilibrium of the screw is most easily derived from the principle of work done. Assuming no work to be lost by friction, we have

work done by effort = work done by resistance;

$\therefore$ P × circumference of circle it describes = resistance × distance through which it is moved;

$\therefore$ P × circumference it describes = resistance × interval between threads of screw.

181. **Method of Winding the Thread on the Cylinder of a Screw.**

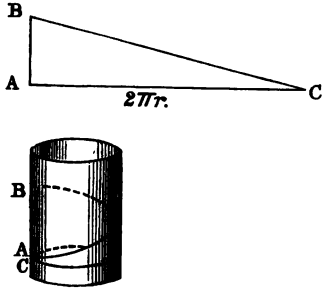

Fig. 56.

Suppose $ABCD$ to be a right-angled triangle cut out of paper. Suppose it twisted round so that its base becomes the circumference of a circle, as shown in Fig. 56. The hypotenuse (BC) will form a helix (*i. e.* curve formed by thread of a screw). The height (AB) of the triangle evidently becomes the distance between two consecutive laps (or threads) of the helix. This is called the *pitch* of the screw.

182. Hunter's Differential Screw

consists of two screws differing slightly in pitch (*i. e.* distance between two consecutive threads). The smaller of these screws is rigidly attached to a platform *R* (Fig. 55), arranged so as to be incapable of turning round with the larger screw *abcd*, which turns with the handles (*P*, *P*). The smaller screw fits into the cylinder of the larger, which forms its nut. When the handles are turned the larger screw is lowered through a distance equal to *its* pitch, while at the same time the smaller screw is raised *relatively to the larger* through a distance equal to *its* pitch. Hence the platform *R* descends a distance = difference of the pitches of the two screws. Hence, by the principle of work done (Part II., Chap. V.),

Effort × circumference of the circle it describes = resistance × difference of the pitches of the two screws.

EXERCISES.

1. The interval between the threads of a screw is one-ninth of an inch, and the diameter of the circle described by the power is 2 feet; find in tons the resistance which a power of 51 tons will sustain.—(Dr. Traill.)

Ans. 15·4 tons.

2. A screw press, the interval between whose threads is $\frac{1}{8}$ inch, is capable of applying a pressure of 11 cwt. when a force of 2 lbs. is applied to its arm. Find the length of the arm, assuming $\pi = \frac{22}{7}$.—(Mr. Purser.)

Ans. $24\frac{1}{2}$ inches.

3. Show how the mechanical advantage to be derived from a screw is deduced from the theory of the inclined plane. If the distance between two consecutive threads of a screw be $\frac{1}{4}$ in., and the length of the power arm 5 ft.; prove that a power of 1 lb. will sustain a weight of 480π lbs.—(Dr. Traill.)

4. The length of the lever of a screw is 21 inches, and the distance between the threads is $\frac{1}{4}$ inch. What power acting perpendicularly to the lever at its extremity will support a weight of 110 lbs.?—(Mr. Cathcart.)

Ans. $\frac{5}{24}$ lbs.

5. If the distance between the threads of a screw be $\frac{1}{2}$ inch, and a force of 10 lbs. be applied at the end of an arm 2 feet long fixed at the centre of the circumference of the screw; what pressure can be produced?—(Mr. Bernard.)

Ans. $12068\frac{4}{7}$ lbs.

6. The tangent of the angle of a screw is $\frac{1}{7}$, the radius of the cylinder is 10 inches, and to the head of the screw a power of 5 lbs. is applied at a radial distance of 2 feet; find the amount of resistance which it can overcome.

Ans. 377 lbs. nearly.

7. In a differential screw the pitches are $\frac{1}{3}$ and $\frac{1}{4}$ of an inch, respectively, and an effort of 10 lbs. is applied at the end of an arm 12 inches long; find the resistance balanced. *Ans.* 9047·8 lbs.

183. The Wedge

is a triangular prism (usually isosceles) made of some hard material, and may be regarded as a movable inclined plane. It is used to separate bodies, or the parts of bodies held together by great forces. This is effected by introducing the edge of the wedge between the parts to be separated, and urging it forwards by pressure, or by blows applied at its back. The relation between the forces acting on the faces of a smooth wedge may be thus obtained:—

Let ABC be a section of a smooth wedge introduced into the cleft YFZ, and let a *pressure* P be employed at X, and perpendicular to CB, to urge the wedge forwards. Let the reactions at Y and Z be Q and R respectively. Then P, Q, and R balance; and since they are perpendicular respectively to the sides a, b, and c of the triangle ABC,

$$\frac{P}{a} = \frac{Q}{b} = \frac{R}{c} \text{ (Art. 59).}$$

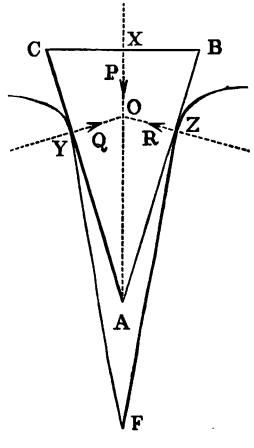

Fig. 57.

These equations furnish us with the relations connecting the forces acting on the faces of a smooth wedge. If the force P be communicated by the blow of a hammer or other instrument, the force so produced is termed an impulsive force, and produces an enormous pressure for a very short time (*Vide* Part II., Chap. III.)

Such cutting instruments as hatchets, knives, chisels, nails, needles, &c., are modifications of the wedge. Wedges are sometimes used to tighten the parts of a structure

EXERCISES.

1. In an isosceles wedge whose vertical angle is 2α, if R be the resistance which each of the equal sides experiences; show that

$$P = 2R \sin \alpha.$$

2. In an isosceles wedge whose base $= a$, and equal sides each equal b; show that $\dfrac{\text{effort}}{\text{whole resistance}^*} = \dfrac{a}{2b}$.

3. What is the ratio of P to R in an isosceles wedge whose base is one-half the length of each of the equal sides? *Ans.* 1 : 2.

4. What is the ratio of the effort to the whole resistance* in an isosceles wedge whose vertical angle is 8°, given $\sin 4° = \cdot 06975$?

Ans. 6975 : 100,000.

5. In an isosceles *rough* wedge, if P be the driving force impressed perpendicular to the back, and R the total resistance on each side when the wedge is about to slip forwards; prove that $P = 2R \sin(\alpha + \lambda)$, where α is the semi-vertical angle of the wedge, and λ the angle of friction between the wedge and body it is employed to split.

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* By whole resistance is here meant the *sum* of the reristances on the two sides of the wedge.

CHAPTER XIV.

III.—THE PULLEY.

184. A pulley is a thin cylinder or wheel with a groove cut in its circumference round which a rope can pass. This wheel is called *the sheaf*, and is capable of turning round an axis which is supported by a piece (or frame) called the *block*.

A pulley is fixed or movable according as its block is fixed or movable. A single fixed pulley *H*, as shown, Fig. 58, gives no mechanical advantage. It, however, secures a change of direction of *P* which may be desirable.

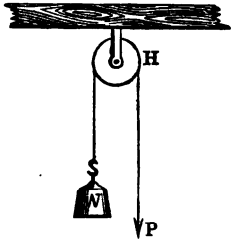

Fig. 58.

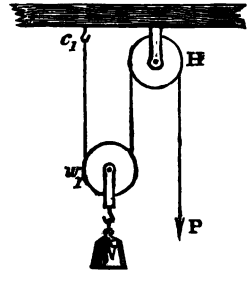

Fig. 59.

185. **Principle of Frictionless Pulley Action.**

The great principle of frictionless pulley action is, that the force acting on any terminated cord employed is the same throughout its length (*cf.* Art. 46).

186. **Single Movable Pulley** (first arrangement).

The adjoining diagram (Fig. 59) represents a single movable pulley (w_1), and a fixed pulley (H). The weight (W) is supported by two parallel tensions (P, P) acting along the two branches of the string formed by w_1.

For equilibrium $2P = W$, or $P = \frac{W}{2}$.

Mechanical Advantage $= \frac{W}{P} = 2$.

If we take in the weight of the movable pulley (w_1), we have

$$2P = W + w_1;$$

therefore mechanical advantage is thereby diminished.

187. **Single Movable Pulley** (second arrangement).

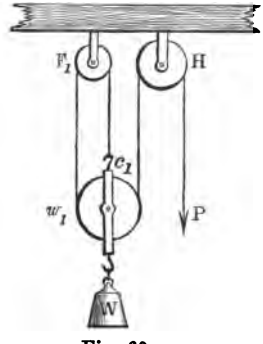

Fig. 60.

The adjoining diagram (Fig. 60) represents another arrangement. Here W being supported by three parallel tensions, we have

$$3P = W; \qquad (1)$$

therefore mechanical advantage

$$= \frac{W}{P} = 3.$$

If we take into account the weight of the movable pulley (w_1), we have

$$3P = W + w_1; \qquad (2)$$

therefore mechanical advantage is thus diminished.

EXERCISES.

I. A man weighing 12 stones forces up a weight of 3 stones by means of a cord passing over a fixed pulley H (Fig. 58), under which he stands. Determine the man's pressure on the ground, and the pressure sustained by the fixed beam. *Ans.* (1) 9 stones; (2) 6 stones.

2. A man standing underneath the pulley H (Fig. 59) in the first arrangement of the single movable pulley, sustains a weight of 12 stones, by means of a rope which he grasps in his hand. Determine the pressure on his hand, and the whole pressure experienced by the fixed beam, supposing the movable pulley to have no weight. *Ans.* (1) 6 stones; (2) 18 stones.

3. In Ex. 2, if the arrangement be that of Fig. 60, determine the same pressures. *Ans.* (1) 4 stones; (2) 16 stones.

4. If in Ex. 2 the weight raised be that of the man himself, together with a movable platform on which he stands, and which is rigidly attached to the movable pulley; determine the same pressures.

Ans. (1) 4 stones; (2) 12 stones.

5. If in Ex. 4 we use the 2nd arrangement of the movable pulley, Fig. 60; determine the pressures. *Ans.* (1) 3 stones; (2) 12 stones.

6. Show that if the two branches of the rope determined by the movable pulley (1st arrangement, Fig. 59) be each inclined to the vertical at an angle θ; then $2P\cos\theta = W$ if we neglect the weight of the movable pulley.

188. **Systems of Pulleys.**—We shall consider three different arrangements of pulleys which we number in the same way as Professor G. M. Minchin.*

189. **First System of Pulleys in which the same String passes round all the Pulleys.**

In this system the pulleys are arranged in two blocks, as shown in Fig. 61, or as in Smeaton's pulley (Art. 192). The cord passes alternately round the pulleys in the upper and lower blocks. The portion of cord which joins any pulley in the upper to any pulley in the lower block is called a *ply*, e. g. XL and YM are plies.

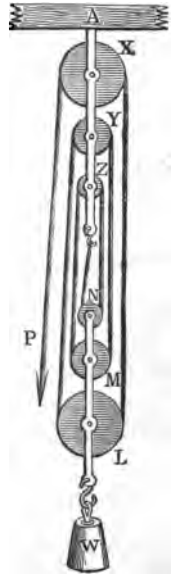

Fig. 61.

The tension on the cord is the same throughout its length (supposing the pulleys smooth), and is equal to the force P applied at its free extremity, which is called the *tackle-fall*.

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* *Treatise on Statics*, third edition. (Messrs. Longmans, Green, & Co., London.)

190. Relation between P and W in First System of Pulleys.

Let n = number of plies at the lower block, i. e. *the whole number of pulleys in the upper and lower blocks taken together.*

P = force applied at free extremity of cord,

W = weight to be lifted,

W_b = weight of lower block.

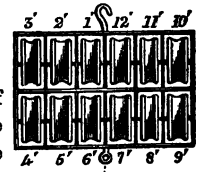
Then $nP = W$, or $= W + W_b$,

according as we neglect or take in the weight of the lower block (W_b), and assuming all the plies to be parallel. This is evident, since W is supported by n parallel tensions each = P.

191. Mechanical Advantage in First System of Pulleys

$$= \frac{W}{P} = n, \text{ or } n - \frac{W_b}{P},$$

according as we neglect or take in the weight of the lower block (W_b). From this it appears that the mechanical advantage is *diminished* by increasing the weight of the movable pulley.



192. Smeaton's Pulley

is represented in the accompanying diagram, Fig. 62. The cord being attached to the upper block at the end of the dotted line, passes

under 1, and over 1′,

under 2, and over 2′,

and so on.

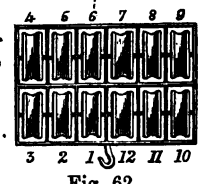


Fig. 62.

The effort is applied at the free end hanging over 12′. This pulley is evidently of the first system.

EXERCISES.

1. Neglecting the weight of the lower block, find the force necessary to support a weight of 120 lbs. in the first system of pulleys, where there are three pulleys in the upper and three in the lower block, and where the fixed end of the cord is attached to the upper block. *Ans.* 20 lbs.

2. If in Ex. 1 there were three pulleys in the upper, and two in the lower block, and if the fixed end of the cord were attached to the lower block; determine the force required to support a weight of 120 lbs.

Ans. 24 lbs.

3. If in Ex. 1 the lower block weigh 30 lbs.; determine the required force when this weight is not neglected. *Ans.* 25 lbs.

4. If in Ex. 2 we take in the weight of the lower block (viz. 30 lbs.); determine the required force. *Ans.* 30 lbs.

5. Describe the system of pulleys in which one string passes round all the pulleys, and find the force required to support a given weight.

What weight can be supported by a force of 2 lbs. by means of a system of 8 pulleys arranged in this way, the weight of the lower block being 1 lb.? —(*The Previous.*) *Ans.* 15 lbs.

6. We are told that the cable by which "Great Paul," 18 tons in weight, was lifted into its place in the Cathedral tower, passed four times through the two blocks of pulleys. From this account give a description of the pulleys, and an estimate of the strength of the cable.—(*The Previous.*)

Leaving out the weight of the lower block,

$$8P = 18 \text{ tons};$$

$$\therefore \quad P = 2\tfrac{1}{4} \text{ tons},$$

which represents the strain which the cable had to sustain throughout.

7. Find what weight a power of 1 cwt. would support by means of two movable and two fixed pulleys, in the system in which a single string passes round all the pulleys, the weight of the block being 1 lb.—(*The Previous.*)

Ans. 447 lbs.

8. In that system of pulleys in which there is only one string, if there are altogether 9 pulleys, and each block of pulleys weighs one pound; what force will be required to support a weight of 100 lbs.?—(*The Previous.*)

Ans. $9\frac{9}{11}$ lbs.

193. Second System of Pulleys in which each Pulley hangs by a Separate String.

We have to consider two cases.

In Case I. (Fig. 63), the ends of the various ropes are all attached to a fixed beam at the points c_1, c_2, c_3, &c.

In Case II. (Fig. 64), the ropes passing over fixed pulleys F_1, F_2, &c., have their ends attached to the movable pulleys at the points c_1, c_2, &c.

194. Relation between P and W in Case I. (Second System.)

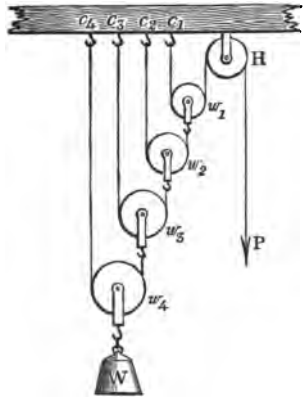


Fig. 63.

The effort or power P is communicated unaltered throughout the first string which goes round w_1, and thereby conveys a tension $= 2P$ to the second string. Thus, *neglecting weights of pulleys*,

$$w_1 \text{ hands on } 2P \qquad = 2P,$$
$$w_2 \quad ,, \quad ,, \quad 2 \times 2P \quad = 2^2P,$$
$$w_n \quad ,, \quad ,, \quad 2 \times 2^{n-1}P = 2^nP.$$

W is supported by this last tension ;

$$\therefore \quad 2^nP = W. \tag{1}$$

Cor. 1. If the weights of the pulleys be taken in account,

$$2^nP = W + 2^{n-1}w_1 + 2^{n-2}w_2 + \&c., + w_n. \tag{2}$$

For effective tension handed on

by $w_1 = 2P - w_1 = 2P - w_1$,

,, $w_2 = 2(2P - w_1) - w_2 = 2^2P - 2w_1 - w_2$,

. . . . . .

,, $w_n = \ldots 2^nP - 2^{n-1}w_1 - \ldots - w_n$;

$\therefore$ $W = 2^nP - 2^{n-1}w_1 - 2^{n-2}w_2 - \&c. - w_n$,

whence equation (2) appears.

Cor. 2.—If the movable pulleys be all of equal weight (w),

$$W = 2^nP - w(2^n - 1). \qquad (3)$$

195. **Relation between P and W in Case II.** (Second System.)

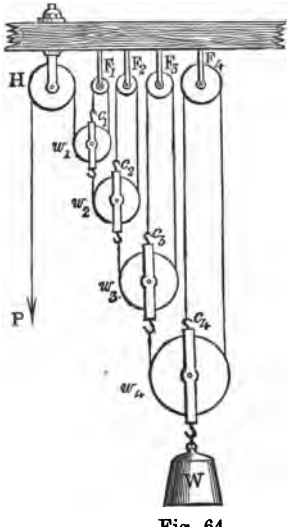


Fig. 64.

Here, *neglecting weights of pulleys*,

w_1 hands on $3P$, $= 3P$

w_2 ,, ,, $3 \times 3P$ $= 3^2P$,

. . . . . .

w_n ,, ,, $3 \times 3^{n-1}P = 3^nP$.

The last tension supports W;

$$\therefore \quad 3^nP = W. \qquad (1)$$

Cor. 1.—If the weights of the movable pulleys be taken into account, the relation between P and W is given by the equation

$$3^n P = W + 3^{n-1}w_1 + 3^{n-2}w_2 + \&c. + w_n. \qquad (2)$$

Cor. 2.—If the weights of the movable pulleys be each $= w$, then

$$3^n P = W + w\,\frac{3^n - 1}{2}. \qquad (3)$$

196. Mechanical Advantage in the Second System of Pulleys.

Mechanical advantage $= \frac{W}{P}$

,, ,, $= 2^n$ (Case I.),

and ,, ,, $= 3^n$ (Case II.),

neglecting the weights of the movable pulleys. It is evident from Figs. 63 and 64 that if these weights are taken into account mechanical advantage is diminished, as also appears from the equations (2), Art. 194, and (2), Art. 195.

N.B.—In Arts. 194, 195, and 196, n represents the number of movable pulleys.

EXERCISES.

1. Find the effort necessary to raise 27 cwt. in the second system of pulleys, Cases I. and II., respectively, when there are three movable pulleys supposed without weight.

Ans. In Case I. $P = 3\frac{3}{8}$ cwt.; in Case II. $P = 1$ cwt.

2. Find the mechanical advantage of a system of pulleys of equal weight (w), each hanging by a separate string, one end of which is attached to a fixed support, where P is the effort employed. *Ans.* $2^n - (2^n - 1)\frac{w}{P}$.

3. How many movable pulleys, each weighing 1 lb., must be used in such a system as that of Ex. 2 for a power of 2 lbs. to support a weight of 65 lbs.? If the weight be pulled down through one inch, how much will the power be raised?—(*The Previous.*) *Ans.* (1) 6; (2) 64 inches.

4. Find the power required to support a weight of 10 lbs. when there are four pulleys, each weighing 1 oz., in the system of pulleys in which each pulley hangs by a separate string, and the strings are all parallel.—(*The Previous.*) *Ans.* $10\frac{1}{8}$ ozs.

5. Describe two different systems of pulleys in which the ratio of the power to the weight is 1 to 8, neglecting the weight of the pulleys.—(*The Previous.*)

Ans. (1) When the same string passes round all the pulleys there are 8 plies at the lower block, and therefore 8 pulleys in all. (2) When each pulley hangs by a separate string there are 3 pulleys.

6. In that system of pulleys in which each pulley hangs in the loop of a separate string, if there be four movable pulleys whose weights, beginning from the lowest, are 4, 3, 2, 1 lb., respectively; find what power will support a weight of 54 lbs.—(*The Previous.*) *Ans.* 5 lbs.

7. In Exercise 6, find the effort necessary to support a weight of 38 lbs. *Ans.* 3 lbs.

8. If there be three movable pulleys in the second system of pulleys whose weights, beginning from the top, are 4, 5, and 6 lbs., respectively; find what weight an effort of 1 cwt. will support in Cases I. and II., respectively. *Ans.* Case I. 864 lbs.; Case II. 2967 lbs.

9. If the movable pulleys in Ex. 8 each weigh 4 lbs.; find the weights supported. *Ans.* Case I. 868 lbs.; Case II. 2972 lbs.

197. Third System of Pulleys in which each String is attached to the Weight.

We have to consider two cases—

Case I. (Fig. 65), in which the rope-ends are attached to W at the points c_1, c_2, c_3, &c.

Case II. (Fig. 66), in which the ropes pass round fixed pulleys F_1, F_2, F_3, &c., and then have their ends attached to the other pulleys.

198. Relation between P and W in the Third System of Pulleys. (Case I., Fig. 65.)

Here, *neglecting the weights of the pulleys,*

$$W = P(2^n - 1). \qquad (1)$$

For w_1 hands on to cord round w_2 a tension $= 2P$,

w_2 ,, ,, ,, w_3 ,, $= 2^2P$,

. . . . . .

w_{n-1} ,, ,, ,, w_n ,, $= 2^{n-1}P$,

w_n ,, the supporting beam a pressure $= 2^nP$;

therefore pressure on beam $= 2^n P$.

But ,, ,, $= P + W$;

$\therefore \quad W + P = 2^n P;$

$\therefore \quad W = (2^n - 1) P.$ (1)

This elegant solution is due to Mr. A. F. Dixon, and does not involve the summation of a geometrical series like the ordinary solution.

Cor. 1.—If the weights of the movable pulleys be taken into account,

$$W = P(2^n - 1) + w_1 (2^{n-1} - 1) + w_2 (2^{n-2} - 1) + \&c. + w_{n-1}. \quad (2)$$

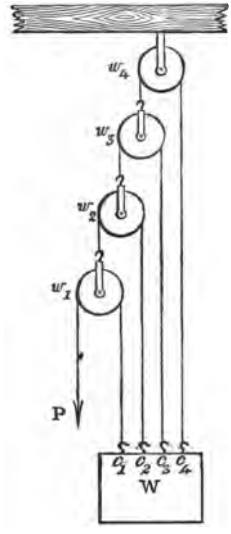


Fig. 65.

For w_1 hands on to cord round w_2 a tension $= 2P$,

w_2 ,, ,, w_3 ,, $= 2^2 P + w_1$,

. . . . . .

w_{n-1} ,, ,, w_n ,, $= 2^{n-1} P + 2^{n-2} w_1 + \&c. + w_{n-2}$,

w_n ,, the supporting beam a pressure $= 2^n P + 2^{n-1} w_1 + \&c. + 2w_{n-1}$;

$\therefore$ pressure on supporting beam $= 2^n P + 2^{n-1} w_1 + \&c. + 2w_{n-1}$.

But ,, ,, ,, $= P + w_1 + w_2 + w_{n-1}$.

Equating these two values of pressure, equation (2) appears.

N.B.—Observe that w_n, which is *not* a movable pulley, does not enter equation (2).

Cor. 2.—If $w_1 = w_2 = w_3$, &c., $= w$,

$$W = P(2^n - 1) + w(2^n - 1 - n). \qquad (3)$$

199. **Relation between P and W in the Third System of Pulleys.** (Case II., Fig. 66).

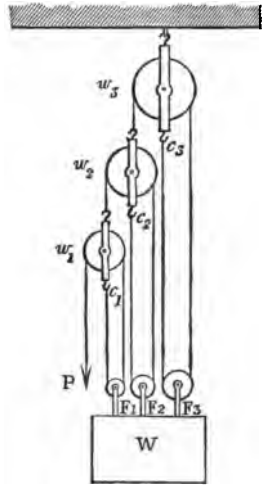


Fig. 66.

Here, *neglecting the weights of the pulleys*,

$$W = (3^n - 1)P. \qquad (1)$$

For w_1 hands on to cord round w_2 a tension $= 3P$,

w_2 ,, ,, ,, w_3 ,, $= 3^2P$,

w_3 ,, ,, ,, w_4 ,, $= 3^3P$,

w_n ,, the supporting beam a pressure $= 3^nP$;

$\therefore$ pressure on supporting beam $= 3^nP$.

But ,, ,, ,, ,, $= P + W$;

$\therefore$ $W + P = 3^nP$;

$\therefore$ $W = (3^n - 1)P$. (1)

Cor. 1.—If the weights of the movable pulleys be taken into account,

$$W = P(3^n - 1) + w_1(3^{n-1} - 1) + \&c.\ 3 + 2w_{n-1}. \qquad (2)$$

N.B.—Observe that w_n, which is *not* a movable pulley, does not enter equation (2).

Cor. 2.—If the weights of the movable pulleys be each equal to w,

$$W = P(3^n - 1) + \frac{3^n - 1 - 2n}{2} w. \qquad (3)$$

200. **Mechanical Advantage in the Third System of Pulleys**

$$= \frac{W}{P} = 2^n - 1 \text{ (Case 1.)}$$

$$= 3^n - 1 \text{ (Case II.)},$$

neglecting the weights of the movable pulleys.

If the weights of the movable pulleys be taken into account, it is evident that they *increase* the mechanical advantage in both cases.

N.B.—Observe that n in Arts. 198, 199, and 200 is the number of pulleys *not* attached to W, and that w_n is *not* a movable pulley.

EXERCISES.

1. In the third system of pulleys, what weight will be supported by a force of 10 lbs., when four distinct cords are employed acting in parallel vertical directions? *Ans.* Case I. 150 lbs.; Case II. 800 lbs.

2. In the third system of pulleys, what weight will be supported by a force of 10 lbs. when there are four *movable pulleys* and the strings are parallel? *Ans.* Case I. 310 lbs.; Case II. 2420 lbs.

3. In Ex. 2, if the movable pulleys each weigh 1 lb., what weight will be supported when the weights of the movable pulleys are taken into account? *Ans.* Case I. 284 lbs.; Case II. 2304 lbs.

4. In the third system of pulleys, what force will be required to support 1 ton when there are three movable weightless pulleys with parallel strings? *Ans.* Case I. 149$\frac{1}{3}$ lbs.; Case II. 28 lbs.

5. If in Ex. 4 each pulley weigh 3 lbs.; determine the force required. *Ans.* Case I. 142$\frac{13}{13}$; Case II. 26$\frac{13}{13}$.

6. Find the condition of equilibrium in the system of pulleys in which the strings are all parallel and attached to the weight. Find the power required to support a weight of 11 lbs., when there are four pulleys each weighing one ounce.—(*The Previous.*) *Ans.* 11 ozs.

7. Find the condition of equilibrium in the system of pulleys in which the strings are all parallel, and one end of each is attached to the lower block. If there be four movable pulleys each weighing 1 lb., and the weight supported be 3 cwt.; find the power.—(*The Previous.*) *Ans.* 10 lbs.

General Exercises.

8. Describe two different systems of pulleys in which the ratio of the power to the weight is 1 to 8, neglecting the weights of the pulleys.—(*The Previous.*)

Ans. (1) System 1 when there are 8 plies at the lower block.
(2) System 2, Case I., with three movable pulleys.

9. Describe two different systems of pulleys in which the ratio of the power to the weight is 1 to 7, neglecting the weights of the pulleys.—(*The Previous.*)

Ans. (1) System 1 with 7 plies at lower block. (2) System 3, Case I., where all the strings are parallel and attached to the weight, and when there are two movable and one fixed pulley, *i.e.* 3 pulleys in all.

10. What power must a man weighing 150 lbs. exert to lift himself by a pair of pulley-blocks, each containing two wheels?—(*College of Preceptors.*) *Ans.* 50 lbs.

11. Make careful sketches of—(α) a system of (weightless) pulleys in which 1 lb. balances 32 lbs., and (β) a system of (weightless) pulleys in which 1 lb. balances 15 lbs., taking care in each case that the number of pulleys is the least possible.—(*London Matriculation.*)

Ans. (1) 2nd system with 5 movable pulleys. (2) 3rd system with 3 movable pulleys.

201. The Differential Pulley

is a modification of the differential wheel and axle, in which the axles are made so short as to receive one lap only of the chain which passes round them in grooves constructed for the purpose. The ends of this chain, instead of being rigidly attached to the axles, as in the differential wheel and axle, are joined together, forming a loop as shown in the diagram (Fig. 67).

The effort or power is applied to that side of the loop cP which comes from the larger axle. The wheels are furnished with projections which, by fitting into the links of the chain, prevent it from slipping.

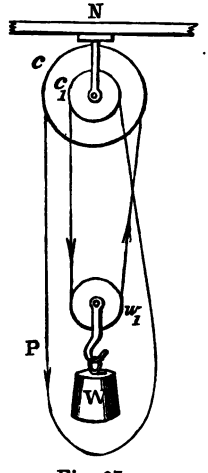


Fig. 67.

Relation between effort (P) and resistance (W).—It is evident that while P moves through the circumference (c) of the larger wheel, W is raised through a distance

$$= \frac{c - c_1}{2};$$

$$\therefore \quad Pc = \frac{W(c - c_1)}{2}. \quad \text{(Part II., Chap. V.)}$$

EXERCISES.

1. Let c and c_1 = 24 and 18 inches, respectively; find the weight raised by an effort of 12 lbs. *Ans.* 96 lbs.

2. In Ex. 1, find the power required to raise 1 cwt. *Ans.* 14 lbs.

NOTE.

To what Order of Lever does the Oar belong?

In a Paper published in the *Philosophical Magazine* for January, 1887, the Rev. T. K. Abbott, F.T.C.D., raises the above question, and proposes to show "that the vulgar conception of the oar *as a lever of the first order** is correct."

As stated in Art. 152, Part I., the oar is commonly regarded (*i. e.* by writers on mechanics) *as a lever of the second kind.** The object of this note is to reconcile these apparently conflicting statements. It will be convenient to state at once the results arrived at in the subsequent investigation. They are these—1°. The oar must be regarded as a lever of the *second order* if the resistance acting at the rowlock be understood to include, not only the *external resistance* to the boat's motion, due to the action of the fluid in which the boat floats, but also *the reaction engendered by the person of the oarsman* when he pulls the oar. 2°. If we consider *only* the resistance offered by the fluid to the boat's motion, it will be found that *this resistance* is related to the effort employed by the rower at the handle of the oar *in exactly the same way as if this resistance acted at the blade of the oar*, and as if the rowlock were the fulcrum, i. e. *practically as if the oar were a lever of the first order*. These results may be thus obtained:—To simplify matters, let us suppose the boat to be propelled by two perfectly similar oars, one at each side of the boat, on which equal efforts (EE) are exerted by the rower. Let the part of each oar outside the rowlock at each side be "b", and the part inside the rowlock "a", so that the whole length of each oar is $a + b$. Let $2R$ be the resistance to the boat's motion due to the fluid. This we may consider to be equivalent to an *external resistance* R acting at each

* The italics are mine.—(I. W.)

rowlock. Let us suppose the boat to be on the point of moving forwards, *due south*, owing to the effort of the rower. Then consider—

1°. *Equilibrium of rower.*—He is acted on by the oars by a force $= 2E$ (northwards) along medial line of boat, and therefore by the boat itself, by a force $= 2E$ (southwards) along same line. Therefore, again, rower acts on boat with a force $= 2E$ (northwards) along medial line.

2°. *Equilibrium of boat.*—Here we have boat's motion opposed by $2R$ (resistance of fluid), and by $2E$ (resistance due to effort); therefore in all by a resistance $= 2R + 2E$; therefore resistance at each rowlock $= R + E$.

3°. *Equilibrium of oars alone.*—This requires equal northward forces, each $= R + E$ (acting at rowlocks). From one oar by taking moments about its blade-end, we obtain

$$(R + E)\, b = E(a + b);$$

whence
$$Rb = Ea, \qquad (1)$$

i. e. the actual resistance to the motion of the boat due to fluid (viz. R) is connected with the rower's effort *as though the oar were a lever of the first order.*

4°. *Equilibrium of whole system consisting of rower, boat, and oars.*—Let ϕ, ϕ be the forces due to the water acting on the blades of the two oars. These two forces are in equilibrium with the fluid resistance, viz. $2R$;

therefore $2\phi = 2R$;

therefore $\phi = R$.

Hence, by (1) $\phi b = Ea.$ (2)

From equations (1) and (2) it follows that the oar, considered as an instrument to overcome the resistance (R) of the fluid, *acts exactly as if it were a lever of the first order*, with the resistance of the fluid acting at the blade of the oar and having the rowlock for fulcrum.

Whether this result might have been *a priori* predicted from the circumstance that the rowlock is a fixed point relatively to the rower, the author leaves for others to determine.

EXAMINATION PAPERS

PROPOSED IN TRINITY COLLEGE, DUBLIN.

I.

JUNIOR SOPHISTERS (TRINITY, 1872).

MATHEMATICAL PHYSICS.

DR. TARLETON.

1. Find the magnitude of a force required to draw a weight of 10 lbs. up a smooth inclined plane, which makes an angle of 30° with the horizon, the force being parallel to the plane. *Ans.* 5 lbs.

2. If the plane in the last question be rough, and the coefficient of friction be $\frac{1}{\sqrt{3}}$, what will be the magnitude of the force? *Ans.* 10 lbs.

3. Assuming that the resultant of two equal forces meeting at a point may be represented by the diagonal of the parallelogram whose sides represent the component forces, prove that the same mode of representation holds good when one component is double the other.

Ans. Part. II., Appendix.

4. Prove that the true weight of any body is a geometric mean between its apparent weights in the two scales of a balance having unequal arms.

Ans. Moments about fulcrum.

5. If two twists situated in the same plane and in opposite directions have equal moments, prove that they are in equilibrium.

Ans. Part. I., Art. 86.

6. A stone is thrown with a velocity of 30 feet per second in a direction perpendicular to a railway train, which is moving with a velocity of 40 feet per second; find the velocity with which the stone enters a window in the train. *Ans.* 50 feet per second.

7. A force of 10 lbs. produces in one second a velocity of 16 feet per second in a given body; find the weight of the body. *Ans.* 20 lbs.

8. Prove that the velocity acquired by a body in running down a smooth inclined plane is equal to that acquired in falling through the corresponding height. *Ans.* Part II., Chap. IV.

9. A weight Q resting on a rough horizontal plane is drawn along by a weight P attached to Q by a horizontal cord without weight passing over a smooth pulley; if μ denote the coefficient of friction, find the tension of the cord.

Ans. $T = \frac{PQ(1+\mu)}{P+Q}$.

II.

Junior Sophisters (Trinity 1876).

Mechanics.

MR. WILLIAMSON.

1. A mass of 100 lbs. is placed on a smooth plane of 45° inclination, and supported by a cord parallel to the plane; find the tension of the cord.

Ans. 70·71 lbs.

2. Give a geometrical construction for determining the magnitude and direction of the resultant of any number of forces whose directions meet in a point. *Ans.* Part I., Art. 61.

3. A uniform bar, 12 feet long and 112 lbs. weight, is supported on two props respectively situated 1 foot and 4 feet distant from the ends of the bar; find the pressures on the props. *Ans.* 32 lbs. and 80 lbs.

4. Find the condition for equilibrium in the wheel and axle from the principle of constancy of work, neglecting friction.

Ans. $\frac{\text{effort}}{\text{resistance}} = \frac{\text{radius of axle}}{\text{radius of wheel}}$.—(Part II.)

5. Forces of 11 lbs., 12 lbs., and 13 lbs., respectively, act on a point at angles of 120°; find the magnitude and direction of their resultant.

Ans. $\sqrt{3}$ lbs., and perpendicular to force 12.

6. If a uniform pressure of 1 lb. generate in one second a velocity of 3 feet, find the weight of the mass moved. *Ans.* $1\frac{1}{2}$ oz.

7. In the case of a uniform force, write down the equation which connects s, f, and t; and state what these letters represent respectively.—*Ans.* $s = \frac{1}{2}ft^2$.—Part II., Chap. II.

8. If a body be projected vertically with a given velocity in a vacuum, find after what time it will return to the point of projection.

Ans. $\frac{2v_0}{g}$ where v_0 = velocity of projection.

9. If masses of 9 and 11 lbs. be connected, as in Atwood's machine, find the velocity acquired by either weight in 3 seconds.

Ans. $\frac{3g}{10}$ feet per second.

10. If the length of a second's pendulum be increased by one-tenth, how many vibrations will it make in an hour? *Ans.* $3428\frac{4}{7}$.

III.

Junior Sophisters (Trinity, 1880).

Mechanics.

MR. PANTON.

1. A weight of 20 lbs. is held up by two strings, at right angles to each other, attached at their point of junction to the weight, and one of the strings makes an angle of 60° with the vertical. Calculate the tensions on the strings. *Ans.* 10 lbs., and $10\sqrt{3}$ lbs.

2. What force parallel to the horizon will sustain a weight of 250 lbs. on a smooth plane inclined to the horizon at an angle of 60°?

Ans. $250\sqrt{3}$ lbs.

3. If the plane be rough (coefficient of friction = $\frac{1}{3}$), what force parallel to the plane will just suffice to draw the weight up the plane?

Ans. 258·16 lbs.

4. A uniform bar weighs 15 lbs., and is 10 feet long; if weights of 20 lbs. and 11 lbs. be suspended from its extremities, find the point on which it will balance. *Ans.* $4\frac{1}{17}$ feet from 20 end.

5. Three weights are placed at the angular points of a right-angled isosceles triangle, that at the right angle being 1 lb., and each of those at the base angles being 2 lbs.; calculate the distance from the vertex of the centre of gravity of the three weights, each of the equal sides of the triangle being 10 feet. *Ans.* $4\sqrt{2}$ feet from vertex.

6. Find the relation of the power to the resistance for equilibrium of the wheel and axle.—Part I., Art. 165.

7. Find the height a body must fall under the action of gravity to acquire a velocity of one mile per minute. *Ans.* 121 feet.

8. Find the time a heavy body will take to run down a smooth inclined plane, 288 feet in length, making an angle of 30° with the horizon.
Ans. 6 seconds.

9. A railway carriage, weighing 8 tons, moving at the rate of 25 miles an hour, describes a portion of a circle whose radius is a quarter of a mile; calculate the centrifugal force in tons. *Ans.* $\frac{11}{216}$ ton.

10. Describe the construction and use of Atwood's machine, and find the dynamical force of the system.—Part. II., Chap. IV.

IV.

Senior Freshmen (Trinity, 1884).

Statics.

DR. TRAILL.

1. From two points 5 feet apart on a horizontal line two strings are attached to a weight of 30 lbs., their lengths being 3 and 4 feet respectively. Find the tension of each string. *Ans.* 24 lbs. and 18 lbs.

2. Prove that the moments of any two forces round any point on their resultant are equal and opposite.—Part I., Art. 69.

3. If two weights balance each other on the arms of a straight lever in any position, prove that they will still balance each other if the lever be turned through any angle.

Ans. Resultant still passes through fulcrum.—Part I., Art. 73.

4. If a shopman sell to his customers alternately from each scale of a false balance, show that he will be the looser after an even number of sales.

Ans. $a^2 + b^2 > 2ab$, where a and b are arms of false balance, therefore, &c.

5. Show how the mechanical advantage to be derived from a screw is deduced from the theory of the inclined plane. If the distance between two consecutive threads of a screw be $\frac{1}{4}$ in., and the length of the power arm 5 feet; prove that a power of 1 lb. will sustain a weight of 480π lbs.—Part. I., Art. 179.

6. If a man support a weight equal to his own, by means of three movable pulleys arranged according to the first system;* find his pressure on the floor, supposing him to be pulling downwards by a string passing over a fixed pulley.

Ans. $\frac{7W}{8}$, where W = man's weight, and where each movable pulley hangs by a separate string attached to a fixed beam.

7. Show how to find the centre of gravity of the perimeter of a triangle. —Part. I., Art. 101.

8. If the pressure on the fulcrum of a lever be 27 lbs., and if the difference of the weights at its extremities be 3 lbs.; find the weights and the ratio of the arms of the lever. *Ans.* (1) 15 lbs. and 12 lbs. (2) $\frac{4}{5}$.

9. If the force required to overcome friction upon a horizontal railroad be 10 lbs. per ton, find the force necessary to preserve the same speed up an incline of 1 in 21. *Ans.* $116\frac{2}{3}$ lbs. per ton.

10. If R be the resultant of two forces P and Q, and if S be the resultant of R and P, prove that the resultant of S and Q will be equal to $2R$.

Ans. Use Ex. 15, Part I., p. 18.

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* Dr. Traill's first system differs from mine.—(I. W.)

V.

Supplemental Little-go (Hilary, 1886).

Mechanics.

MR. F. PURSER.

1. Two forces of 2 lbs. and 1 lb. respectively act at a point, their directions making an angle of 120°. Find the value of the resultant, and the angles it makes with the component forces.

Ans. $R = \sqrt{3}$, inclined 90° to 1, and 30° to 2.

2. A uniform beam 12 feet long, and weighing 6 cwt., rests on two horizontal supports at its extremities, an additional weight of 3 cwt. being attached to a point 4 feet from one end. Find the stresses on the supports.

Ans. 4 lbs. and 5 lbs.

3. A weight is supported on a smooth inclined plane by a rope, one end of which is attached to it, while the other, passing over a smooth pulley at the top of the plane, carries a second weight hanging freely. If the first weight be double the second, find the inclination of the plane.

Ans. $i = 30°$.

4. Define accurately the centre of gravity, and show how to determine it experimentally for a plane board of any form.—Part I., Arts. 94 and 106.

5. A screw press, the interval of whose threads is $\frac{1}{2}$ inch., is capable of applying a pressure of 11 cwt. when a force of 2 lbs. is applied to its arm. Find the length of arm, assuming $\pi = \frac{22}{7}$. *Ans.* 24$\frac{1}{2}$ inches.

6. State accurately Newton's second law of motion, and explain the terms 'change,' 'quantity of motion,' which occur in it.—Part. II., Chap. III.

7. A weight of 1 ton, moving on a horizontal line of rails, at the rate of 4$\frac{1}{2}$ miles an hour, is, by the action of a constant resistance, reduced in a quarter of an hour to a speed of 4 miles an hour. Find the value of the resistance in pounds, assuming, as usual, $g = 32$. *Ans.* 51$\frac{1}{4}$ lbs.

8. Show that a body moving down a smooth inclined plane from rest will attain the same velocity after descending the same vertical height, whatever be the inclination of the plane. *Ans.* Part II., Chap. IV.

9. A weight of 20 lbs. is whirled round in a circle of 2 feet in radius, the pull on the string being 1 lb. Find its velocity.

Ans. $\frac{4}{\sqrt{5}}$ feet per second.

10. Explain what is meant by the isochronism of a simple pendulum, and give the formula connecting the length of such a pendulum, the value of gravity at the place, and the time of vibration.

Ans. Part II. $T = \pi\sqrt{\frac{l}{g}}$.

VI.

Junior Sophisters (Supplementalists) (Trinity, 1886).

Mechanics.

DR. TARLETON.

1. Two parallel forces, in the same direction, acting on a rigid body, are equivalent to a single force. How does the proof of this proposition fail if the body be not rigid?

Ans. For then forces cannot be considered to act at any point indifferently along their lines of action.—(Part I., Art. 36.)

2. On a smooth inclined plane, whose inclination to the horizon is i, a weight W is placed, and supported by three strings making angles A, B, C with the inclined plane. The strings pass over smooth pulleys, and have weights of 9, 8, and 10 lbs. attached to them. If $\sin i = \frac{1}{20}$, $\cos A = \frac{1}{3}$, $\cos B = \frac{1}{4}$, $\cos C = \frac{1}{5}$; find the magnitude of W in order that there should be equilibrium. *Ans.* 140 lbs.

3. The arms of a faulty balance, which hangs evenly when empty, are 5 inches and 6 inches long; a body placed in the scale attached to the long arm appears to weigh 12 lbs.; what is its true weight? *Ans.* $14\frac{2}{5}$ lbs.

4. The line of direction of a force F, which passes through a point O, makes angles α and β with two straight lines OA and OB, meeting at O; what are the components of F along OA and OB?

Ans. Component along $OA = \frac{F \sin \beta}{\sin(\alpha + \beta)}$.

,, along $OB = \frac{F \sin \alpha}{\sin(\alpha + \beta)}$.

5. Two equal forces P, acting at the same point, are inclined to each other at an angls θ; determine the magnitude of their resultant.

Ans. $R = 2P \sin \frac{\theta}{2}$.

6. How far must a body fall to acquire a velocity of one mile per minute?

Ans. 121 feet.

7. A piece of falling wood, which weighs 20 oz., is penetrated when its velocity is 40 feet per second by a bullet discharged vertically upwards, and moving with a velocity of 1600 feet per second. The bullet remains in the wood, which it reduces to rest; what is its weight? *Ans.* $\frac{1}{2}$ oz.

8. A mass A of 15 lbs., resting on a smooth horizontal table, is attached by a string, whose weight is negligible, passing over the edge of the table to another mass of 5 lbs. which hangs vertically. What is the velocity of A at the end of one second? *Ans.* 8 feet per second.

9. Two bodies are dropped successively from the same point with an interval of $\frac{1}{4}$ of a second. When will the distance between them be 8 feet?

Ans. $1\frac{1}{8}$ seconds after first body is dropped.

10. A particle moves in the inside of a smooth tube, which forms a vertical circle of radius a; what must be the velocity of the particle at the lowest point in order that it should just reach the highest? *Ans.* $v = 2\sqrt{ga}$.

For the last five of the questions above Senior Freshmen are to substitute the following:—

6. If the moments of two couples lying in the same plane be equal in magnitude, show that the forces of the couples coincide in direction with the sides of a parallelogram, and are proportional to them in magnitude.

Ans. Part I., Art. 86.

7. Hence prove, that if the moments be opposite in direction, and the couples act on the same rigid body, they equilibrate.

Ans. Part I., Art. 86.

8. A body A weighing 50 lbs. is placed on a rough horizontal plane: a string fastened to A passes over a smooth pulley at the edge of the plane, and is attached to a weight B, which is gradually increased. When B is 30 lbs. A begins to move: what is the value of the coefficient of friction?

Ans. $\frac{3}{5}$.

9. The components of a force F along two directions OA and OB at right angles to each other are $\sqrt{192}$ and 8 lbs.; find the angle which the direction of F makes with OA. *Ans.* 30°.

10. A string, whose weight is negligible, passes round a smooth pulley, and is in equilibrium: why are the tensions at the two ends of the string equal? *Ans.* Otherwise motion would ensue.

VII.

Junior Sophisters Supplementalizing, Senior Freshmen Examination (Hiliary, 1887).

Mechanics.

MR. CATHCART.

1. If s feet be described in t seconds by a particle having an initial velocity u, write down the value of the acceleration, supposed uniform.

Ans. $f = \frac{2(s - ut)}{t^2}$.

2. If feet and seconds be taken as units, determine the distance described in 11 seconds by a particle starting with a velocity 59, and subject to a uniform retardation 6. *Ans.* 286 feet.

3. If a train weighing 100 tons be reduced to rest from a velocity of 20 miles per hour in 100 yards, find the force of resistance in tons.

Ans. $1\frac{49}{81}$ tons.

4. Prove that if a body be thrown in a horizontal direction from the top of a precipice, its depth at any instant is proportional to the square of its horizontal distance from the vertical through the starting-point.

Ans. $\frac{\text{depth}}{(\text{horizontal distance})^2} = \frac{g}{2v^2}$, where v = horizontal velocity of projection.

5. State accurately the three axioms or laws of motion.—Part II., Chap. III.

6. At what point can a rod 8 feet long, weighing 1 lb. per foot, and having weights of 3 and 5 lbs. fastened on at the two ends, be supported? *Ans.* 6 inches from middle of rod.

7. Write down the magnitude of the resultant of two forces, P and Q, which act at right angles upon any point, and the tangent of the angle its direction makes with that of P.

Ans. $R = \sqrt{P^2 + Q^2}$; $\tan \alpha = \frac{Q}{P}$, where R = resultant, and α = angle it makes with P.

8. To which kind of lever does the oar belong?—Part I., Note at end.

9. If the force of friction, when a body weighing 11 lbs. is just about to slide down a plane inclined at 30° to the horizon, be 5 lbs., what is the coefficient of friction?

Ans. $\mu = \frac{10}{11\sqrt{3}}$.

10. If it be assumed that the direction of the resultant of two forces, acting on a point, is the diagonal of the parallelogram included by them, show that the diagonal represents the resultant also in magnitude.—Part II., Appendix.

VIII.

Senior Freshmen (Michaelmas), 1887.

Mechanics.

DR. TARLETON.

1. If three forces acting at the same point are in equilibrium, prove that they are proportional to, and in the same direction as, the sides of a triangle. —Part I., Art. 57.

2. A force F is equivalent to two forces X and Y at right angles to each other, and acting at the same point. If A be the angle between the directions of F and X, express the magnitude of X and Y in terms of F and A.

Ans. $X = F \cos A$; $Y = F \sin A$.

3. Determine the position of the centre of gravity of the area included between the circumferences of two non-concentric circles whose radii are a and b, and of which one lies entirely inside the other.

Ans. If D = distance between centres of circles, the centre of gravity is at a distance $= \frac{b^2 D}{a^2 - b^2}$ from centre of larger circle on line joining centres of circles.

4. A smooth hemisphere rests with its plane face on the ground; show that there is only one point on the hemisphere at which a heavy particle can be placed so as to be in equilibrium.

Ans. Point vertically above centre of plane face.

5. If the hemisphere be rough, μ being the coefficient of friction, determine the boundary of the positions of equilibrium of the particle.

Ans. A small circle subtending $\angle = 2\angle$ of friction at centre of plane face.

6. How far must a body fall in order to acquire a velocity of 200 feet per second? *Ans.* 625 feet.

7. Prove that the velocity acquired in sliding down a smooth inclined plane is the same as that acquired in falling through the corresponding height. *Ans.* Part II., Chap. IV.

8. A body weighing 5 lbs., and moving on a smooth horizontal table, swings round a fixed point, to which it is attached by a string 9 feet long, with a velocity of 24 feet per second; find the tension of the string.

Ans. 10 lbs.

9. A stone falling from an embankment strikes a train which is moving with the same velocity as the stone; at what angle will the stone impinge against the roof of the train? *Ans.* 45°.

10. The time occupied by a body in sliding down a smooth inclined plane is n times the time occupied by another body in falling through the corresponding height; find the inclination of the plane.

Ans. $i = \sin^{-1}\frac{1}{n}$.

IX.

Senior Freshmen (Trinity, 1888).

Mechanics.

DR. TARLETON.

1. A weight of 60 lbs. is supported by two strings which are respectively, 3 feet and 4 feet long, and are fastened to two points in the same horizontal line, at such a distance apart that the strings are perpendicular to each other; find the tension of each string. *Ans.* 36 lbs. and 48 lbs.

2. Find the horizontal force necessary to support a weight of 1 lb. on a smooth plane which rises 3 in 5. *Ans.* $\frac{3}{5}$ lb.

3. What is the radius of a wheel, if a power of 3 ozs. is just able to move a weight of 12 ozs. which hangs from the axle, the radius of the axle being 2 inches? *Ans.* 8 inches.

4. A uniform beam 3 feet long, the weight of which is 10 lbs., is supported in a horizontal position across a rail, with 4 lbs. hanging from one end and 13 lbs. from the other; find the point of the beam in contact with the rail.
Ans. 6 inches from middle of beam.

5. Prove that the centre of gravity of a homogeneous plate in the form of a parallelogram is the point of intersection of the diagonals.
Ans. Since diagonals bisect each other ∴ &c., Part I., Art. 99.

6. A force F makes an angle θ with the line OA; find the forces, X in direction of OA and Y perpendicular to OA, which are equivalent to F, being given $F = 52$ lbs., $\cos\theta = \frac{12}{13}$. *Ans.* $X = 48$ lbs.; $Y = 20$ lbs.

7. From a uniform circular plate another plate, likewise circular and having for its diameter the radius of the first circle, is cut away; find the centre of gravity of the remainder.
Ans. $\frac{r}{3}$ from centre of larger plate, when r = radius of smaller plate.

8. Two couples having equal and opposite moments lie in the same plane. If the forces of the one couple be not parallel to those of the other, show that the four forces may be represented in magnitude and direction by the sides of a parallelogram.—Part I., Art. 86.

9. A cord, whose length is $2l$, is fastened at A and B lying in the same horizontal line at a distance $2a$ from each other; a smooth ring on the cord supports a weight W; find the tension of the cord in terms of W, l, and a.
Ans. $T = \dfrac{Wl}{2\sqrt{l^2 - a^2}}$.

10. A force P, making an angle θ with a rough inclined plane, is just able to draw up a body whose weight is W. If i be the inclination of the plane, find the coefficient of friction. *Ans.* $\dfrac{P\cos\theta - W\sin i}{W\cos i - P\sin\theta}$.

For the last four of the foregoing Questions, Junior Sophisters supplementalizing the Final Senior Freshmen Examination are to substitute the following:—

1. With what velocity must a body be projected vertically upwards in order that it may ascend to a height of 49 feet?

Ans. 56 feet per second.

2. A body falls freely for 6 seconds; what space will it describe during the last second? *Ans.* 176 feet.

3. A person is running in a straight line with a uniform velocity. In what direction should he throw up a ball in order that it may fall back into his hand? *Ans.* Vertically upwards.

4. What weight hanging vertically downwards will draw a weight of $3\frac{1}{2}$ lbs. across a perfectly smooth table 8 feet wide in 2 seconds?

Ans. 7 ozs.

X.

Senior Freshmen (Michaelmas, 1888).

Mechanics.

MR. RUSSELL.

Statics.

[*Supplementalists may confine their attention to the Questions on Statics.*]

1. Three forces act in a plane, and are in equilibrium. How are they situated? *Ans.* Part I., Art. 57.

2. A piece of timber weighing one ton is supported by two ropes, which are inclined to the vertical on opposite sides at angles of 60° each; find the tensions on the ropes. *Ans.* 1 ton.

3. Two forces of 16 lbs. and 14 lbs. respectively act at an angle of 120°; find the resultant. *Ans.* 15·1 lbs. If $\angle = 60°$, $R = 26$.

4. A uniform rod, 10 inches long, and weighing 5 lbs., has weights of 10 lbs. and 15 lbs. respectively hanging from its ends; at what distance from the centre will it balance? *Ans.* $\frac{2}{3}$ inch.

5. Prove that the moment of the resultant of two forces about any point is equal to the sum of the moments of the forces about the same point.—Part I., Art. 71.